The General Triangle Is Unique

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Some acute-angled triangles are special, e.g. right-angled or isosceles triangles. Some are not of this kind, but, without measuring angles, look as if they are. In that sense, there is exactly one general triangle. This well-known fact[1] is proven here formally.

theory GeneralTriangle
imports Complex-Main
begin

1 Type definitions

Since we are only considering acute-angled triangles, we define angles as numbers from the real interval $[0\ldots90]$.

abbreviation angles $\equiv \{ \, x:\text{real} \, . \, 0 \leq x \land x \leq 90 \, \}$

Triangles are represented as lists consisting of exactly three angles which add up to 180°. As we consider triangles up to similarity, we assume the angles to be given in ascending order.

Isabelle expects us to prove that the type is not empty, which we do by an example.

definition triangle $=$
\{ $l$. $l \in$ lists angles $\land$
  length $l = 3$ $\land$
  listsum $l = 180$ $\land$
\}
\begin{verbatim}

typedef triangle = triangle

unfolding triangle-def
apply (rule-tac x = [45,45,90] in exI)
apply auto
done

For convenience, the following lemma gives us easy access to the three angles of a
triangle and their properties.

\begin{lemma} unfold-triangle: \end{lemma}
\begin{proof}
\begin{verbatim}
obtain a b c where Rep-triangle t = [a,b,c]
and a ∈ angles
and b ∈ angles
and c ∈ angles
and a + b + c = 180
and a ≤ b
and b ≤ c
\end{verbatim}
\end{proof}
\end{verbatim}

\section*{2 Property definitions}

Two angles can be considered too similar if they differ by less than 15°. This number
is obtained heuristically by a field experiment with an 11th grade class and was chosen
that statistically, 99% will consider the angles as different.

\begin{definition} similar-angle :: real ⇒ real ⇒ bool (infix \sim 50)
where similar-angle x y = (abs (x - y) < 15) \end{definition}

The usual definitions of right-angled and isosceles, using the just introduced similarity
for comparison of angles.

\begin{definition} right-angled \end{definition}
where \( \text{right-angled } l = (\exists x \in \text{set} (\text{Rep-triangle } l) . x \sim 90) \)

definition isosceles
where \( \text{isosceles } l = ((\text{Rep-triangle } l) ! 0 \sim (\text{Rep-triangle } l) ! 1 \lor (\text{Rep-triangle } l) ! 1 \sim (\text{Rep-triangle } l) ! (\text{Suc } 1)) \)

A triangle is special if it is isosceles or right-angled, and general if not. Equilateral triangle are isosceles and thus not mentioned on their own here.

definition special
where \( \text{special } t = (\text{isosceles } t \lor \text{right-angled } t) \)

definition general
where \( \text{general } t = (\neg \text{special } t) \)

3 The Theorem

definition general
where \( \exists ! t . \text{general } t \)

The proof proceeds in two steps: There is a general triangle, and it is unique. For the first step we give the triangle (angles 45°, 60° and 75°), show that it is a triangle and that it is general.

proof
have is-t [simp]: \([45, 60, 75] \in \text{triangle}\) by (auto simp add: triangle-def)
show general (Abs-triangle \([45,60,75]\)) (is general \(?t\))
by (auto simp add:general-def special-def isosceles-def right-angled-def
Abs-triangle-inverse similar-angle-def)

next
For the second step, we give names to the three angles and successively find upper bounds to them.

fix \( t \)
obtain \( a b c \) where
\( abc: \text{Rep-triangle } t = [a,b,c] \)
and \( a \in \text{angles} \) and \( b \in \text{angles} \) and \( c \in \text{angles} \)
and \( a \leq b \) and \( b \leq c \)
and \( a + b + c = 180 \)
by (rule unfold-triangle)

assume general \( t \)
hence ni: \( \neg \text{isosceles } t \) and nra: \( \neg \text{right-angled } t \)
by (auto simp add: general-def special-def)

have \( \neg c \sim 90 \) using nra abc
by (auto simp add:right-angled-def)
hence \( c \leq 75 \) using \( c \in \text{angles} \)
by (auto simp add:similar-angle-def)

have \( \neg b \sim c \) using ni abc
by (auto simp add:isosceles-def)
hence \( b \leq 60 \) using \( b \leq c \) and \( c \leq 75 \)

3
have \( \lnot a \sim b \) using \( \text{ni abc} \)
by (auto simp add: similar-angle-def)

have \( a \leq 45 \) using \( (a \leq b) \text{ and } (b \leq 60) \)
by (auto simp add: isosceles-def)

The upper bound is actually the value, or we would not have a triangle

have \( \lnot (c < 75 \lor b < 60 \lor a < 45) \)
proof
assume \( c < 75 \lor b < 60 \lor a < 45 \)
hence \( a + b + c < 180 \) using \( (c \leq 75) \) \( (b \leq 60) \) \( (a \leq 45) \)
and \( (a \in \text{angles}) \) \( (b \in \text{angles}) \) \( (c \in \text{angles}) \)
by auto
thus False using \( (a + b + c = 180) \) by auto
qed

hence \( c = 75 \) and \( b = 60 \) and \( a = 45 \)
using \( (c \leq 75) \) \( (b \leq 60) \) \( (a \leq 45) \)
by auto

And this concludes the proof.

hence \( \text{Abs-triangle } (\text{Rep-triangle } t) = \text{Abs-triangle } [45, 60, 75] \)
using \( \text{abc} \) by simp
thus \( t = \text{Abs-triangle } [45, 60, 75] \) by (simp add: Rep-triangle-inverse)
qed

end

References