Lifting Definition Option*

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Abstract

We implemented a command, lift-definition-option, which can be used to easily generate elements of a restricted type \( \{ x :: \text{'}a \cdot P \} \), provided the definition is of the form \( \lambda y_1 \ldots y_n. \text{if check } y_1 \ldots y_n \text{ then Some } (\text{generate } y_1 \ldots y_n :: \text{'}a) \text{ else None } \) and \( \text{check } y_1 \ldots y_n = \Rightarrow P \ldots (\text{generate } y_1 \ldots y_n) \) can be proven.

In principle, such a definition is also directly possible using one invocation of lift-definition. However, then this definition will not be suitable for code-generation. To this end, we automated a more complex construction of Joachim Breitner which is amenable for code-generation, and where the test \( \text{check } y_1 \ldots y_n \) will only be performed once. In the automation, one auxiliary type is created, and Isabelle’s lifting- and transfer-package is invoked several times.

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theory Lifting-Definition-Option-Explanation
imports
   Lifting-Definition-Option
   Rat
begin

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1 Introduction

Often algorithms expect that their input satisfies some specific property $P$. For example, some algorithms require lists as input which have to be sorted, or numbers which have to be positive, or programs which have to be well-typed, etc. Here, there are at least two approaches how one can reason about these algorithms.

The first approach is to guard all soundness properties of the algorithm by the additional precondition $P$ input. So, as an example, a binary search algorithm might take arbitrary lists as input, but only if the input list is sorted, then the result of the search is meaningful. Whereas for binary search, this approach is reasonable, there might be problems that the restriction on the input is even crucial for actually defining the algorithm, since without the restriction the algorithm might be non-terminating, and thus, cannot be easily defined using Isabelle’s function-package [4]. As an example, consider an approximation algorithm for $\sqrt{x}$, where the algorithm should stop once the current approximation $y$ satisfies $|x^2 - y^2| < \delta$. Imagine now, that $\delta$ is a negative number.

To this end, in the second approach the idea is to declare restricted types, so that the algorithms are only invoked with inputs which satisfy the property $P$. For example, using Isabelle’s lifting- and transfer-package [3], one can easily define a dedicated type for positive numbers, and a function which accesses the internal number, which is then guaranteed to be positive.

```
typedef 'a pos-num = { x :: 'a :: linordered-field. x > 0 } morphisms num pos
by (rule exI[of - 1], auto)
```

Using these restricted types, it is often possible to define the desired algorithms where non-termination because of invalid inputs is no longer possible, and where soundness properties can be stated without additional preconditions. E.g., for the approximation algorithm, one takes $\delta$ as input, and the approximation stops, once $|x^2 - y^2| < num \delta$ is satisfied.

One question in the second approach is how to actually generate elements of the restricted type. Although one can perform the following definition,

```
lift-definition create-pos-1 :: 'a :: linordered-field ⇒ 'a pos-num option is
λ x. if x > 0 then Some x else None
by auto
```

the problem is that corresponding defining equation `create-pos-1 ?x = map-option os (if (0::'?a) < ?x then Some ?x else None)` is not amenable for code-generation, as it uses the abstraction function `pos` in a way which is not admitted for code-equations [1, 2].
To overcome this problem, Joachim Breitner proposed the following workaround\(^1\), which requires an additional type definition, and some auxiliary definitions, to in the end define \(\text{create-pos}\) in a way that is amenable for code-generation.

\[
\text{typedef} \ 'a \text{ num-bit} = \{ (x :: 'a :: \text{linordered-field}, b). b \rightarrow x > 0 \} \text{ by auto}
\]

\[
\text{setup-lifting type-definition-num-bit}
\]

\[
\text{lift-definition num-bit-bit :: ('a :: \text{linordered-field}) num-bit \Rightarrow bool} \text{ is snd}.
\]

\[
\text{lift-definition num-bit-num :: ('a :: \text{linordered-field}) num-bit \Rightarrow 'a pos-num} \text{ is}
\]

\[
\lambda (x,b). \text{if } b \text{ then } x \text{ else } 42 \text{ by auto}
\]

\[
\text{lift-definition num-bit :: 'a :: \text{linordered-field} \Rightarrow 'a num-bit} \text{ is}
\]

\[
\lambda x. \text{if } x > 0 \text{ then } (x, \text{True}) \text{ else } (42, \text{False}) \text{ by auto}
\]

\[
\text{definition create-pos-2 :: 'a :: \text{linordered-field} \Rightarrow 'a pos-num} \text{ option where}
\]

\[
\text{create-pos-2 } x \equiv \text{let } nb = \text{num-bit } x
\]

\[
\text{in if num-bit-bit nb then Some (num-bit-num nb) else None}
\]

\[
\text{lemma create-pos-2: create-pos-2 } x = \text{Some } p \Rightarrow \text{num } p = x
\]

\[
\text{unfolding create-pos-2-def Let-def by (transfer, simp split: if-splits)}
\]

\[
\text{export-code create-pos-2 in Haskell}
\]

Breitner’s construction has the advantage that the invariant \((0::'a) < x\) only has to be evaluated once (when invoking \(\text{num-bit}\)). Hence, the construction allows to create data for types with invariants in an efficient, executable, and canonical way.

In this AFP entry we now turned this canonical way into a dedicated method (\textit{lift-definition-option}) which automatically generates the types and auxiliary functions of Breitner’s construction. As a result it suffices to write:

\[
\text{lift-definition-option create-pos :: 'a :: \text{linordered-field} \Rightarrow 'a pos-num} \text{ option is}
\]

\[
\lambda x :: 'a. \text{if } x > 0 \text{ then Some } x \text{ else None}
\]

\[
\text{by auto}
\]

Afterwards, we can directly generate code.

\[
\text{export-code create-pos in Haskell}
\]

Moreover, we automatically generate two soundness theorems, that the generated number is the intended one: \(\text{create-pos } ?x = (\text{if } (0::?'a) < ?x \text{ then Some (pos ?x) else None})\) and \(\text{create-pos } ?x = \text{Some } ?r \Rightarrow \text{num } ?r = ?x\). Here, the morphisms from the type-definitions reappear, i.e., \text{num} and \text{pos} in the example.

\(^1\text{http://stackoverflow.com/questions/16273812/working-with-isabelles-code-generator-data-refinement-and-higher-order-functio}
2 Usage and limitations

The command \texttt{lift-definition-option} is useful to generate elements of some restricted type (say \texttt{'restricted}) which has been defined as \{x. P x\} for some property \(P\) of type \texttt{'base \Rightarrow bool}. It expects three arguments, namely

- The name of the definition, e.g., \texttt{create-pos}.
- The type of the definition, which must be of the form \texttt{'a1 \Rightarrow 'a2 \Rightarrow 'a-dots \Rightarrow 'a-n \Rightarrow 'restricted option}.
- The right-hand side of the definition which must be of the shape \(\lambda x_1 x_2 x\text{-}dots x\text{-}n. \text{if check } x_1 x_2 x\text{-}dots x\text{-}n \text{ then Some (generate } x_1 x_2 x\text{-}dots x\text{-}n\text{) else None}\) where \texttt{generate} is of type \texttt{'a1 \Rightarrow 'a2 \Rightarrow 'a-dots \Rightarrow 'a-n \Rightarrow 'base option}.

After providing the three arguments, a proof of \texttt{check x_1 x_2 x\text{-}dots x\text{-}n \Rightarrow P (generate x_1 x_2 x\text{-}dots x\text{-}n)} has to be provided. Then, code-equations will be derived and registered, and in addition two soundness theorems are generated. These are accessible under the names \texttt{def-name} and \texttt{def-name-Some}, provided that the lifting definition uses \texttt{def-name} as first argument.

Note, that \(P\) is automatically extracted from the type-definition of \texttt{'restricted}. Similarly, the default value (42 in the \texttt{'a pos-num-example}) is generated automatically.

To mention a further limitation besides the strict syntactic structure for the right-hand side, it is sometimes required, to add explicit type-annotations in the right-hand-side and the selector, e.g., the \texttt{'a} in \(\lambda x. \text{if } (0::'a) < x \text{ then Some } x \text{ else None}\).

end

theory Lifting-Definition-Option-Examples
imports
  Lifting-Definition-Option
begin

3 Examples

3.1 A simple restricted type without type-parameters

typedef restricted = \{ i :: int. i mod 2 = 0\} morphisms base restricted
  by (intro exI[of - 4]) auto
setup-lifting type-definition-restricted

Let us start with just using a sufficient criterion for testing for even numbers, without actually generating them, i.e., where the generator is just the identity function.
lift-definition-option restricted-of-simple :: int ⇒ restricted option is
λ x :: int. if x ∈ {0, 2, 4, 6} then Some x else None
proof –
  fix x :: int
  assume x ∈ {0, 2, 4, 6}
  thus x mod 2 = 0 by auto
qed

We can also take several input arguments for the test, and generate a more complex value.

lift-definition-option restricted-of-many-args :: nat ⇒ int ⇒ bool ⇒ restricted option is
λ x y (b :: bool). if int x + y = 5 then Some ((int x + 1) * (y + 1)) else None
proof –
  fix x y and b :: bool
  assume int x + y = 5
  from arg-cong[OF this, of λ x. x mod 2]
  have int x mod 2 = 1 ∨ y mod 2 = 1 by (auto simp: field-simps)
  hence (int x + 1) mod 2 = 0 ∨ (y + 1) mod 2 = 0 by presburger
  thus ((int x+1) * (y + 1)) mod 2 = 0 by auto
qed

No problem to use type parameters.

lift-definition-option restricted-of-poly :: 'b list ⇒ restricted option is
λ xs :: 'b list. if length xs = 2 then Some (int (length (xs))) else None
proof –
  fix xs :: 'b list
  assume length xs = 2
  thus int (length xs) mod 2 = 0 by auto
qed

3.2 Examples with type-parameters in the restricted type.
typedef 'f restrictedf = { xs :: 'f list. length xs < 3 } morphisms basef restrictedf

by (intro exI[of - Nil]) auto
setup-lifting type-definition-restrictedf

It does not matter, if we take the same or different type-parameters in the lift-definition.

lift-definition-option test1 :: 'g ⇒ nat ⇒ 'g restrictedf option is
λ (e :: 'g) x. if x < 2 then Some (replicate x e) else None
proof –
  fix e :: 'g and x
  show x < 2 ⟹ length (replicate x e) < 3 by auto
qed

lift-definition-option test2 :: 'f ⇒ nat ⇒ 'f restrictedf option is
λ (e :: 'f) x. if x < 2 then Some (replicate x e) else None
proof –
  fix e :: 'f and x
  show x < 2 ⇒ length (replicate x e) < 3 by auto
qed

Tests with multiple type-parameters.

typedef ('a,'f) restr = { (xs :: 'a list,ys :: 'f list) . length xs = length ys}
morphisms base' restr
  by (rule exI[of - ([] , [])], auto)
setup-lifting type-definition-restr

lift-definition-option restr-of-pair :: 'g ⇒ 'e list ⇒ nat ⇒ nat ⇒ ('e,nat) restr
  option is
  λ (z :: 'g) (xs :: 'e list) (y :: nat) n. if length xs = n then Some (xs,replicate n y) else None
proof –
  fix xs :: 'e list and y :: nat and n
  assume length xs = n
  thus case (xs, replicate n y) of (xs, ys) ⇒ length xs = length ys
    by auto
qed

3.3 Example from IsaFoR/CeTA

An argument filter is a mapping π from n-ary function symbols into lists of positions, i.e., where each position is between 0 and n-1. In IsaFoR, (Isabelle’s Formalization of Rewriting) and CeTA [5], the corresponding certifier for term rewriting related properties, this is modelled as follows, where a partial argument filter in a map is extended to a full one by means of a default filter.

typedef 'f af = { (π :: 'f × nat ⇒ nat list). (∀ f n. set (π (f,n)) ⊆ {0 ..< n})}
morphisms af Abs-af by (rule exI[of - λ -. []], auto)
setup-lifting type-definition-af

type-synonym 'f af-impl = ('f × nat) × nat list

fun fun-of-map-fun :: ('a ⇒ 'b option) ⇒ ('a ⇒ 'b) where
  fun-of-map-fun m f a = (case m a of Some b ⇒ b | None ⇒ f a)

lift-definition-option af-of :: 'f af-impl ⇒ 'f af option is
  λ s :: 'f af-impl. if (∀ fidx ∈ set s. (∀ i ∈ set (snd fidx). i < snd (fst fidx)))
  then Some (fun-of-map-fun (map-of s) (λ (f,n). ![0 ..< n])) else None
proof (intro allI)
  fix pi :: 'f af-impl and f n
  let ?pi = fun-of-map-fun (map-of pi) (λ (f,n). ![0 ..< n])
  assume *: ∀ fidx∈set pi. ∀ i∈set (snd fidx). i < snd (fst fidx)

show set (?pi (f, n)) ⊆ {0..<n}

proof (cases map-of pi (f,n))
  case (Some idx)
  from map-of-SomeD[OF Some] Some
  have ?pi (f,n) = idx ((f, n), idx) ∈ set pi by auto
  with * Some show ?thesis by auto
qed auto

qed

3.4 Code generation tests and derived theorems

export-code
  restricted-of-many-args
  restricted-of-simple
  restricted-of-poly
  test1
  test2
  restr-of-pair
  af-of
in Haskell

thm
  restricted-of-many-args-Some restricted-of-many-args
  restricted-of-simple-Some restricted-of-simple
  restricted-of-poly-Some restricted-of-poly
  test1-Some test1
  test2-Some test2
  restr-of-pair-Some restr-of-pair
  af-of-Some af-of
end

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References


