

Category Theory to Yoneda's Lemma

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This development proves Yoneda's lemma and aims to be readable by humans. It only defines what is needed for the lemma: categories, functors and natural transformations. Limits, adjunctions and other important concepts are not included.

There is no explanation or discussion in this document. See [O'K04] for this and a survey of category theory formalisations.

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1 Categories

```
theory Cat
imports FuncSet
begin
```

1.1 Definitions

```
record ('o, 'a) category =
  ob :: 'o set (Ob1 70)
  ar :: 'a set (Ar1 70)
  dom :: 'a ⇒ 'o (Dom1 - [81] 70)
  cod :: 'a ⇒ 'o (Cod1 - [81] 70)
  id :: 'o ⇒ 'a (Id1 - [81] 80)
  comp :: 'a ⇒ 'a ⇒ 'a (infixl · 60)
```

definition

```
hom :: [('o, 'a, 'm) category-scheme, 'o, 'o] ⇒ 'a set (Hom1 - -) where
hom CC A B = { f. f ∈ ar CC & dom CC f = A & cod CC f = B }
```

locale category =

```
fixes CC (structure)
assumes dom-object [intro]:
  f ∈ Ar ⇒ Dom f ∈ Ob
and cod-object [intro]:
  f ∈ Ar ⇒ Cod f ∈ Ob
and id-left [simp]:
  f ∈ Ar ⇒ Id (Cod f) · f = f
and id-right [simp]:
  f ∈ Ar ⇒ f · Id (Dom f) = f
and id-hom [intro]:
  A ∈ Ob ⇒ Id A ∈ Hom A A
and comp-types [intro]:
  ⋀ A B C. (comp CC) : (Hom B C) → (Hom A B) → (Hom A C)
and comp-associative [simp]:
  f ∈ Ar ⇒ g ∈ Ar ⇒ h ∈ Ar
  ⇒ Cod h = Dom g ⇒ Cod g = Dom f
  ⇒ f · (g · h) = (f · g) · h
```

1.2 Lemmas

lemma (in category) homI:

```
assumes f ∈ Ar and Dom f = A and Cod f = B
shows f ∈ Hom A B
using assms by (auto simp add: hom-def)
```

lemma (in category) homE:

```
assumes A ∈ Ob and B ∈ Ob and f ∈ Hom A B
shows Dom f = A and Cod f = B
```

proof –

show $Dom\ f = A$ **using** *assms* **by** (*simp add: hom-def*)
show $Cod\ f = B$ **using** *assms* **by** (*simp add: hom-def*)
qed

lemma (*in category*) *id-arrow* [*intro*]:
assumes $A \in Ob$
shows $Id\ A \in Ar$
proof –
from $\langle A \in Ob \rangle$ **have** $Id\ A \in Hom\ A\ A$ **by** (*rule id-hom*)
thus $Id\ A \in Ar$ **by** (*simp add: hom-def*)
qed

lemma (*in category*) *id-dom-cod*:
assumes $A \in Ob$
shows $Dom\ (Id\ A) = A$ **and** $Cod\ (Id\ A) = A$
proof –
from $\langle A \in Ob \rangle$ **have** $1: Id\ A \in Hom\ A\ A$..
then show $Dom\ (Id\ A) = A$ **and** $Cod\ (Id\ A) = A$
by (*simp-all add: hom-def*)
qed

lemma (*in category*) *compI* [*intro*]:
assumes $f: f \in Ar$ **and** $g: g \in Ar$ **and** $Cod\ f = Dom\ g$
shows $g \cdot f \in Ar$
and $Dom\ (g \cdot f) = Dom\ f$
and $Cod\ (g \cdot f) = Cod\ g$
proof –
have $f \in Hom\ (Dom\ f)\ (Cod\ f)$ **using** *f* **by** (*simp add: hom-def*)
with $\langle Cod\ f = Dom\ g \rangle$ **have** *f-homset*: $f \in Hom\ (Dom\ f)\ (Dom\ g)$ **by** *simp*
have *g-homset*: $g \in Hom\ (Dom\ g)\ (Cod\ g)$ **using** *g* **by** (*simp add: hom-def*)
have $(op\ \cdot) : Hom\ (Dom\ g)\ (Cod\ g) \rightarrow Hom\ (Dom\ f)\ (Dom\ g) \rightarrow Hom\ (Dom\ f)\ (Cod\ g)$..
from *this* **and** *g-homset*
have $(op\ \cdot)\ g \in Hom\ (Dom\ f)\ (Dom\ g) \rightarrow Hom\ (Dom\ f)\ (Cod\ g)$
by (*rule funcset-mem*)
from *this* **and** *f-homset*
have *gf-homset*: $g \cdot f \in Hom\ (Dom\ f)\ (Cod\ g)$
by (*rule funcset-mem*)
thus $g \cdot f \in Ar$
by (*simp add: hom-def*)
from *gf-homset* **show** $Dom\ (g \cdot f) = Dom\ f$ **and** $Cod\ (g \cdot f) = Cod\ g$
by (*simp-all add: hom-def*)
qed

end

2 Set is a Category

```
theory SetCat
imports Cat
begin
```

2.1 Definitions

```
record 'c set-arrow =
  set-dom :: 'c set
  set-func :: 'c  $\Rightarrow$  'c
  set-cod :: 'c set
```

definition

```
set-arrow :: ['c set, 'c set-arrow]  $\Rightarrow$  bool where
set-arrow U f  $\longleftrightarrow$  set-dom f  $\subseteq$  U & set-cod f  $\subseteq$  U
  & (set-func f): (set-dom f)  $\rightarrow$  (set-cod f)
  & set-func f  $\in$  extensional (set-dom f)
```

definition

```
set-id :: ['c set, 'c set]  $\Rightarrow$  'c set-arrow where
set-id U = ( $\lambda s \in Pow\ U. (set-dom=s, set-func=\lambda x \in s. x, set-cod=s)$ )
```

definition

```
set-comp :: ['c set-arrow, 'c set-arrow]  $\Rightarrow$  'c set-arrow (infix  $\odot$  70) where
set-comp g f =
  (
    set-dom = set-dom f,
    set-func = compose (set-dom f) (set-func g) (set-func f),
    set-cod = set-cod g
  )
```

definition

```
set-cat :: 'c set  $\Rightarrow$  ('c set, 'c set-arrow) category where
set-cat U =
  (
    ob = Pow U,
    ar = {f. set-arrow U f},
    dom = set-dom,
    cod = set-cod,
    id = set-id U,
    comp = set-comp
  )
```

2.2 Simple Rules and Lemmas

```
lemma set-objectI [intro]: A  $\subseteq$  U  $\Longrightarrow$  A  $\in$  ob (set-cat U)
by (simp add: set-cat-def)
```

```
lemma set-objectE [intro]: A  $\in$  ob (set-cat U)  $\Longrightarrow$  A  $\subseteq$  U
```

by (*simp add: set-cat-def*)

lemma *set-homI* [*intro*]:

assumes $A \subseteq U$

and $B \subseteq U$

and $f : A \rightarrow B$

and $f \in \text{extensional } A$

shows $(\setminus \text{set-dom}=A, \text{set-func}=f, \text{set-cod}=B) \in \text{hom } (\text{set-cat } U) A B$

using *assms* by (*simp add: set-cat-def hom-def set-arrow-def*)

lemma *set-dom* [*simp*]: $\text{dom } (\text{set-cat } U) f = \text{set-dom } f$

by (*simp add: set-cat-def*)

lemma *set-cod* [*simp*]: $\text{cod } (\text{set-cat } U) f = \text{set-cod } f$

by (*simp add: set-cat-def*)

lemma *set-id* [*simp*]: $\text{id } (\text{set-cat } U) A = \text{set-id } U A$

by (*simp add: set-cat-def*)

lemma *set-comp* [*simp*]: $\text{comp } (\text{set-cat } U) g f = g \odot f$

by (*simp add: set-cat-def*)

lemma *set-dom-cod-object-subset* [*intro*]:

assumes $f : f \in \text{ar } (\text{set-cat } U)$

shows $\text{dom } (\text{set-cat } U) f \in \text{ob } (\text{set-cat } U)$

and $\text{cod } (\text{set-cat } U) f \in \text{ob } (\text{set-cat } U)$

and $\text{set-cod } f \subseteq U$

and $\text{set-dom } f \subseteq U$

proof–

note [*simp*] = *set-cat-def set-arrow-def*

have $\text{dom } (\text{set-cat } U) f = \text{set-dom } f$ **using** f **by** *simp*

also show $\dots \subseteq U$ **using** f **by** *simp*

finally show $\text{dom } (\text{set-cat } U) f \in \text{ob } (\text{set-cat } U)$..

have $\text{cod } (\text{set-cat } U) f = \text{set-cod } f$ **using** f **by** *simp*

also show $\dots \subseteq U$ **using** f **by** *simp*

finally show $\text{cod } (\text{set-cat } U) f \in \text{ob } (\text{set-cat } U)$..

qed

In this context, $f \in \text{hom } A B$ is quite a strong claim.

lemma *set-homE* [*intro*]:

assumes $f : f \in \text{hom } (\text{set-cat } U) A B$

shows $A \subseteq U$

and $B \subseteq U$

and $\text{set-dom } f = A$

and $\text{set-func } f : A \rightarrow B$

and $\text{set-cod } f = B$

proof–

have $1 : f \in \text{ar } (\text{set-cat } U)$

```

    using f by (simp add: hom-def set-cat-def)
  show 2: set-dom f = A
    using f by (simp add: set-cat-def hom-def set-arrow-def)
  from 1 have set-dom f  $\subseteq$  U ..
  thus A  $\subseteq$  U by (simp add: 2)
  show 3: set-cod f = B
    using f by (simp add: set-cat-def hom-def set-arrow-def)
  from 1 have set-cod f  $\subseteq$  U ..
  thus B  $\subseteq$  U by (simp add: 3)
  have set-func f  $\in$  (set-dom f)  $\rightarrow$  (set-cod f)
    using f by (auto simp add: set-cat-def hom-def set-arrow-def)
  thus set-func f  $\in$  A  $\rightarrow$  B
    by (simp add: 2 3)
qed

```

2.3 Set is a Category

lemma *set-id-left*:

```

  assumes f: f  $\in$  ar (set-cat U)
  shows set-id U (set-cod f)  $\odot$  f = f
proof -
  from  $\langle f \in \text{ar } (\text{set-cat } U) \rangle$  have set-cod f  $\subseteq$  U ..
  hence 1: set-id U (set-cod f)  $\odot$  f =
    (
      set-dom=set-dom f,
      set-func=compose (set-dom f) ( $\lambda x \in \text{set-cod f. } x$ ) (set-func f),
      set-cod=set-cod f
    )
    using f by (simp add: set-comp-def set-id-def)
  have 2: compose (set-dom f) ( $\lambda x \in \text{set-cod f. } x$ ) (set-func f) = set-func f
  proof (rule extensionalityI)
    show compose (set-dom f) ( $\lambda x \in \text{set-cod f. } x$ ) (set-func f)  $\in$  extensional (set-dom
  f)
      by (rule compose-extensional)
    show set-func f  $\in$  extensional (set-dom f)
      using f by (simp add: set-cat-def set-arrow-def)
    fix x
    assume x-in-dom: x  $\in$  set-dom f
    have f-into-cod: set-func f : (set-dom f)  $\rightarrow$  (set-cod f)
      using f by (simp add: set-cat-def set-arrow-def)
    from f-into-cod and x-in-dom
    have f-x-in-cod: set-func f x  $\in$  set-cod f
      by (rule funcset-mem)
    show compose (set-dom f) ( $\lambda x \in \text{set-cod f. } x$ ) (set-func f) x = set-func f x
      by (simp add: x-in-dom f-x-in-cod compose-def)
  qed
  from 1 have set-id U (set-cod f)  $\odot$  f =
    (
      set-dom=set-dom f,

```

$set\text{-}func = set\text{-}func\ f$,
 $set\text{-}cod = set\text{-}cod\ f$
 \rangle
by (*simp only: 2*)
also have $\dots = f$
by *simp*
finally show *?thesis* .
qed

lemma *set-id-right*:

assumes $f: f \in ar\ (set\text{-}cat\ U)$
shows $f \odot (set\text{-}id\ U\ (set\text{-}dom\ f)) = f$

proof –

from $\langle f \in ar\ (set\text{-}cat\ U) \rangle$ **have** $set\text{-}dom\ f \subseteq U$..

hence $1: f \odot (set\text{-}id\ U\ (set\text{-}dom\ f)) =$

\langle
 $set\text{-}dom = set\text{-}dom\ f$,
 $set\text{-}func = compose\ (set\text{-}dom\ f)\ (set\text{-}func\ f)\ (\lambda x \in set\text{-}dom\ f. x)$,
 $set\text{-}cod = set\text{-}cod\ f$
 \rangle

using f **by** (*simp add: set-comp-def set-id-def*)

have $2: compose\ (set\text{-}dom\ f)\ (set\text{-}func\ f)\ (\lambda x \in set\text{-}dom\ f. x) = set\text{-}func\ f$

proof (*rule extensionalityI*)

show $compose\ (set\text{-}dom\ f)\ (set\text{-}func\ f)\ (\lambda x \in set\text{-}dom\ f. x) \in extensional\ (set\text{-}dom\ f)$

by (*rule compose-extensional*)

show $set\text{-}func\ f \in extensional\ (set\text{-}dom\ f)$

using f **by** (*simp add: set-cat-def set-arrow-def*)

fix x

assume $x\text{-in-dom}: x \in set\text{-}dom\ f$

thus $compose\ (set\text{-}dom\ f)\ (set\text{-}func\ f)\ (\lambda x \in set\text{-}dom\ f. x)\ x = set\text{-}func\ f\ x$

by (*simp add: compose-def*)

qed

from 1 **have** $f \odot (set\text{-}id\ U\ (set\text{-}dom\ f)) =$

\langle
 $set\text{-}dom = set\text{-}dom\ f$,
 $set\text{-}func = set\text{-}func\ f$,
 $set\text{-}cod = set\text{-}cod\ f$
 \rangle

by (*simp only: 2*)

also have $\dots = f$

by *simp*

finally show *?thesis* .

qed

lemma *set-id-hom*:

assumes $A \in ob\ (set\text{-}cat\ U)$

shows $id\ (set\text{-}cat\ U)\ A \in hom\ (set\text{-}cat\ U)\ A\ A$

proof –

from $\langle A \in \text{ob } (\text{set-cat } U) \rangle$ **have** $1: A \subseteq U$..
hence $\text{id } (\text{set-cat } U) A = (\setminus \text{set-dom}=A, \text{set-func}=\lambda x \in A. x, \text{set-cod}=A)$
by $(\text{simp add: set-cat-def set-id-def})$
also have $\dots \in \text{hom } (\text{set-cat } U) A A$
proof (rule set-homI)
show $(\lambda x \in A. x) \in A \rightarrow A$
by $(\text{rule funcsetI, auto})$
show $(\lambda x \in A. x) \in \text{extensional } A$
by $(\text{rule restrict-extensional})$
qed (rule 1, rule 1)
finally show $?thesis$.
qed

lemma *set-comp-types*:

$\text{comp } (\text{set-cat } U) \in \text{hom } (\text{set-cat } U) B C \rightarrow \text{hom } (\text{set-cat } U) A B \rightarrow \text{hom } (\text{set-cat } U) A C$

proof (rule funcsetI)

fix g

assume $g\text{-}BC: g \in \text{hom } (\text{set-cat } U) B C$

hence $\text{comp-cod}: \text{set-cod } g = C$..

show $\text{comp } (\text{set-cat } U) g \in \text{hom } (\text{set-cat } U) A B \rightarrow \text{hom } (\text{set-cat } U) A C$

proof (rule funcsetI)

fix f

assume $f\text{-}AB: f \in \text{hom } (\text{set-cat } U) A B$

hence $\text{comp-dom}: \text{set-dom } f = A$..

show $\text{comp } (\text{set-cat } U) g f \in \text{hom } (\text{set-cat } U) A C$

proof–

have $\text{comp } (\text{set-cat } U) g f =$

$($

$\text{set-dom} = A,$

$\text{set-func} = \text{compose } (\text{set-dom } f) (\text{set-func } g) (\text{set-func } f),$

$\text{set-cod} = C$

$)$

by $(\text{simp add: set-cat-def set-comp-def comp-cod comp-dom})$

also have $\dots \in \text{hom } (\text{set-cat } U) A C$

proof (rule set-homI)

from $f\text{-}AB$ **show** $A \subseteq U$..

from $g\text{-}BC$ **show** $C \subseteq U$..

from $f\text{-}AB$ **have** $fs\text{-}f: \text{set-func } f: A \rightarrow B$..

from $g\text{-}BC$ **have** $fs\text{-}g: \text{set-func } g: B \rightarrow C$..

from $fs\text{-}g$ **and** $fs\text{-}f$

show $\text{compose } (\text{set-dom } f) (\text{set-func } g) (\text{set-func } f) : A \rightarrow C$

by $(\text{simp only: comp-dom}) (\text{rule funcset-compose})$

show $\text{compose } (\text{set-dom } f) (\text{set-func } g) (\text{set-func } f) \in \text{extensional } A$

by $(\text{simp only: comp-dom}) (\text{rule compose-extensional})$

qed

finally show $?thesis$.

qed

qed
qed

We reason explicitly about the function component of the composite arrow, leaving the rest to the simplifier.

lemma *set-comp-associative*:

fixes *f* **and** *g* **and** *h*
assumes *f*: $f \in ar \ (set-cat \ U)$
and *g*: $g \in ar \ (set-cat \ U)$
and *h*: $h \in ar \ (set-cat \ U)$
and *hg*: $cod \ (set-cat \ U) \ h = dom \ (set-cat \ U) \ g$
and *gf*: $cod \ (set-cat \ U) \ g = dom \ (set-cat \ U) \ f$
shows $comp \ (set-cat \ U) \ f \ (comp \ (set-cat \ U) \ g \ h) =$
 $comp \ (set-cat \ U) \ (comp \ (set-cat \ U) \ f \ g) \ h$
proof (*simp add: set-cat-def set-comp-def*)
show $compose \ (set-dom \ h) \ (set-func \ f) \ (compose \ (set-dom \ h) \ (set-func \ g) \ (set-func \ h)) =$
 $compose \ (set-dom \ h) \ (compose \ (set-dom \ g) \ (set-func \ f) \ (set-func \ g)) \ (set-func \ h)$
proof (*rule compose-assoc*)
have *1*: $set-cod \ h = set-dom \ g$ **using** *hg* **by** *simp*
have *2*: $set-cod \ g = set-dom \ f$ **using** *gf* **by** *simp*
show $set-func \ h \in set-dom \ h \rightarrow set-dom \ g$
using *h* **by** (*simp add: set-cat-def set-arrow-def 1*)
show $set-func \ g \in set-dom \ g \rightarrow set-dom \ f$
using *g* **by** (*simp add: set-cat-def set-arrow-def 2*)
show $set-func \ f \in set-dom \ f \rightarrow set-cod \ f$
using *f* **by** (*simp add: set-cat-def set-arrow-def*)
qed
qed

theorem *set-cat-cat*: *category* (*set-cat U*)

proof (*rule category.intro*)

fix *f*
assume *f*: $f \in ar \ (set-cat \ U)$
show $dom \ (set-cat \ U) \ f \in ob \ (set-cat \ U)$ **using** *f* **..**
show $cod \ (set-cat \ U) \ f \in ob \ (set-cat \ U)$ **using** *f* **..**
show $comp \ (set-cat \ U) \ (id \ (set-cat \ U) \ (cod \ (set-cat \ U) \ f)) \ f = f$
using *f* **by** (*simp add: set-id-left*)
show $comp \ (set-cat \ U) \ f \ (id \ (set-cat \ U) \ (dom \ (set-cat \ U) \ f)) = f$
using *f* **by** (*simp add: set-id-right*)
next
fix *A*
assume *A* $\in ob \ (set-cat \ U)$
then show $id \ (set-cat \ U) \ A \in hom \ (set-cat \ U) \ A \ A$
by (*rule set-id-hom*)
next
fix *A* **and** *B* **and** *C*

```

show  $\text{comp } (\text{set-cat } U) \in \text{hom } (\text{set-cat } U) B C \rightarrow \text{hom } (\text{set-cat } U) A B \rightarrow \text{hom}$ 
 $(\text{set-cat } U) A C$ 
  by (rule set-comp-types)
next
  fix  $f$  and  $g$  and  $h$ 
  assume  $f \in \text{ar } (\text{set-cat } U)$ 
    and  $g \in \text{ar } (\text{set-cat } U)$ 
    and  $h \in \text{ar } (\text{set-cat } U)$ 
    and  $\text{cod } (\text{set-cat } U) h = \text{dom } (\text{set-cat } U) g$ 
    and  $\text{cod } (\text{set-cat } U) g = \text{dom } (\text{set-cat } U) f$ 
  then show  $\text{comp } (\text{set-cat } U) f (\text{comp } (\text{set-cat } U) g h) =$ 
 $\text{comp } (\text{set-cat } U) (\text{comp } (\text{set-cat } U) f g) h$ 
    by (rule set-comp-associative)
qed

end

```

3 Functors

```

theory Functors
imports Cat
begin

```

3.1 Definitions

```

record  $(\text{'o1}, \text{'a1}, \text{'o2}, \text{'a2})$  functor =
   $om :: \text{'o1} \Rightarrow \text{'o2}$ 
   $am :: \text{'a1} \Rightarrow \text{'a2}$ 

```

abbreviation

```

 $om\text{-syn } (- \circ [81])$  where
 $F \circ \equiv om F$ 

```

abbreviation

```

 $am\text{-syn } (- \text{a} [81])$  where
 $F \text{a} \equiv am F$ 

```

```

locale two-cats =  $AA$ : category  $AA$  +  $BB$ : category  $BB$ 
  for  $AA$  (structure) and  $BB$  (structure) +
  assumes  $AA = (AA :: (\text{'o1}, \text{'a1}, \text{'m1}) \text{category-scheme})$ 
  assumes  $BB = (BB :: (\text{'o2}, \text{'a2}, \text{'m2}) \text{category-scheme})$ 
  fixes  $\text{preserves-dom} :: (\text{'o1}, \text{'a1}, \text{'o2}, \text{'a2}) \text{functor} \Rightarrow \text{bool}$ 
  and  $\text{preserves-cod} :: (\text{'o1}, \text{'a1}, \text{'o2}, \text{'a2}) \text{functor} \Rightarrow \text{bool}$ 
  and  $\text{preserves-id} :: (\text{'o1}, \text{'a1}, \text{'o2}, \text{'a2}) \text{functor} \Rightarrow \text{bool}$ 
  and  $\text{preserves-comp} :: (\text{'o1}, \text{'a1}, \text{'o2}, \text{'a2}) \text{functor} \Rightarrow \text{bool}$ 
  defines  $\text{preserves-dom } G \equiv$ 
 $\forall f \in Ar_1. G \circ (Dom_1 f) = Dom_2 (G \text{a} f)$ 
  and  $\text{preserves-cod } G \equiv$ 

```

$\forall f \in Ar_1. G_o (Cod_1 f) = Cod_2 (G_a f)$
and *preserves-id* $G \equiv$
 $\forall A \in Ob_1. G_a (Id_1 A) = Id_2 (G_o A)$
and *preserves-comp* $G \equiv$
 $\forall f \in Ar_1. \forall g \in Ar_1. Cod_1 f = Dom_1 g \longrightarrow G_a (g \cdot_1 f) = (G_a g) \cdot_2 (G_a f)$

locale *functor* = *two-cats* +

fixes F (**structure**)

assumes *F-preserves-arrows*: $F_a : Ar_1 \rightarrow Ar_2$

and *F-preserves-objects*: $F_o : Ob_1 \rightarrow Ob_2$

and *F-preserves-dom*: *preserves-dom* F

and *F-preserves-cod*: *preserves-cod* F

and *F-preserves-id*: *preserves-id* F

and *F-preserves-comp*: *preserves-comp* F

notes *F-axioms* = *F-preserves-arrows* *F-preserves-objects* *F-preserves-dom*

F-preserves-cod *F-preserves-id* *F-preserves-comp*

notes *func-pred-defs* = *preserves-dom-def* *preserves-cod-def* *preserves-id-def* *preserves-comp-def*

This gives us nicer notation for asserting that things are functors.

abbreviation

Functor (*Functor* - : - \longrightarrow - [81]) **where**

Functor $F : AA \longrightarrow BB \equiv$ *functor* AA BB F

3.2 Simple Lemmas

For example:

lemma (**in** *functor*) *Functor* $F : AA \longrightarrow BB$..

lemma *functors-preserve-arrows* [*intro*]:

assumes *Functor* $F : AA \longrightarrow BB$

and $f \in ar$ AA

shows $F_a f \in ar$ BB

proof –

from \langle *Functor* $F : AA \longrightarrow BB$ \rangle

have $F_a : ar$ $AA \rightarrow ar$ BB

by (*simp add: functor-def functor-axioms-def*)

from *this* **and** \langle $f \in ar$ AA \rangle

show *?thesis* **by** (*rule funcset-mem*)

qed

lemma (**in** *functor*) *functors-preserve-homsets*:

assumes $1: A \in Ob_1$

and $2: B \in Ob_1$

and $3: f \in Hom_1 A B$

shows $F_a f \in Hom_2 (F_o A) (F_o B)$

proof –

from 3

have $4: f \in Ar$
by (*simp add: hom-def*)
with *F-preserves-arrows*
have $5: F_a f \in Ar_2$
by (*rule funcset-mem*)
from 4 **and** *F-preserves-dom*
have $Dom_2 (F_a f) = F_o (Dom_1 f)$
by (*simp add: preserves-dom-def*)
also from 3 **have** $\dots = F_o A$
by (*simp add: hom-def*)
finally have $6: Dom_2 (F_a f) = F_o A$.
from 4 **and** *F-preserves-cod*
have $Cod_2 (F_a f) = F_o (Cod_1 f)$
by (*simp add: preserves-cod-def*)
also from 3 **have** $\dots = F_o B$
by (*simp add: hom-def*)
finally have $7: Cod_2 (F_a f) = F_o B$.
from 5 **and** 6 **and** 7
show *?thesis*
by (*simp add: hom-def*)
qed

lemma *functors-preserve-objects* [*intro*]:
assumes *Functor F : AA \longrightarrow BB*
and $A \in ob\ AA$
shows $F_o A \in ob\ BB$
proof –
from $\langle Functor\ F : AA \longrightarrow BB \rangle$
have $F_o : ob\ AA \rightarrow ob\ BB$
by (*simp add: functor-def functor-axioms-def*)
from *this* **and** $\langle A \in ob\ AA \rangle$
show *?thesis* **by** (*rule funcset-mem*)
qed

3.3 Identity Functor

definition

id-func :: $(\prime o, \prime a, \prime m)$ *category-scheme* \Rightarrow $(\prime o, \prime a, \prime o, \prime a)$ *functor* **where**
id-func $CC = (\prime om = (\lambda A \in ob\ CC. A), \prime am = (\lambda f \in ar\ CC. f))$

locale *one-cat* = *two-cats* +
assumes *endo: BB = AA*

lemma (**in** *one-cat*) *id-func-preserves-arrows*:
shows $(id-func\ AA)_a : Ar \rightarrow Ar$
by (*unfold id-func-def, rule funcsetI, simp*)

lemma (in *one-cat*) *id-func-preserves-objects*:
 shows $(id\text{-func } AA)_o : Ob \rightarrow Ob$
 by (unfold *id-func-def*, rule *funcsetI*, *simp*)

lemma (in *one-cat*) *id-func-preserves-dom*:
 shows *preserves-dom* (*id-func AA*)
 unfolding *preserves-dom-def* *endo*

proof
 fix *f*
 assume $f : f \in Ar$
 hence *lhs*: $(id\text{-func } AA)_o (Dom f) = Dom f$
 by (*simp add: id-func-def*) *auto*
 have $(id\text{-func } AA)_a f = f$
 using *f* by (*simp add: id-func-def*)
 hence *rhs*: $Dom (id\text{-func } AA)_a f = Dom f$
 by *simp*
 from *lhs* and *rhs* show $(id\text{-func } AA)_o (Dom f) = Dom (id\text{-func } AA)_a f$
 by *simp*
qed

lemma (in *one-cat*) *id-func-preserves-cod*:
preserves-cod (*id-func AA*)
 apply (unfold *preserves-cod-def*, *simp only: endo*)

proof
 fix *f*
 assume $f : f \in Ar$
 hence *lhs*: $(id\text{-func } AA)_o (Cod f) = Cod f$
 by (*simp add: id-func-def*) *auto*
 have $(id\text{-func } AA)_a f = f$
 using *f* by (*simp add: id-func-def*)
 hence *rhs*: $Cod (id\text{-func } AA)_a f = Cod f$
 by *simp*
 from *lhs* and *rhs* show $(id\text{-func } AA)_o (Cod f) = Cod (id\text{-func } AA)_a f$
 by *simp*
qed

lemma (in *one-cat*) *id-func-preserves-id*:
preserves-id (*id-func AA*)
 unfolding *preserves-id-def* *endo*

proof
 fix *A*
 assume $A : A \in Ob$
 hence *lhs*: $(id\text{-func } AA)_a (Id A) = Id A$
 by (*simp add: id-func-def*) *auto*
 have $(id\text{-func } AA)_o A = A$
 using *A* by (*simp add: id-func-def*)
 hence *rhs*: $Id ((id\text{-func } AA)_o A) = Id A$

```

    by simp
  from lhs and rhs show (id-func AA)a (Id A) = Id ((id-func AA)o A)
    by simp
qed

```

```

lemma (in one-cat) id-func-preserves-comp:
  preserves-comp (id-func AA)
unfolding preserves-comp-def endo
proof (intro ballI impI)
  fix f and g
  assume f: f ∈ Ar and g: g ∈ Ar and Cod f = Dom g
  then have g · f ∈ Ar ..
  hence lhs: (id-func AA)a (g · f) = g · f
    by (simp add: id-func-def)
  have id-f: (id-func AA)a f = f
    using f by (simp add: id-func-def)
  have id-g: (id-func AA)a g = g
    using g by (simp add: id-func-def)
  hence rhs: (id-func AA)a g · (id-func AA)a f = g · f
    by (simp add: id-f id-g)
  from lhs and rhs
  show (id-func AA)a (g · f) = (id-func AA)a g · (id-func AA)a f
    by simp
qed

```

```

theorem (in one-cat) id-func-functor:
  Functor (id-func AA) : AA → AA
proof -
  from id-func-preserves-arrows
  and id-func-preserves-objects
  and id-func-preserves-dom
  and id-func-preserves-cod
  and id-func-preserves-id
  and id-func-preserves-comp
  show ?thesis
    by unfold-locales (simp-all add: endo preserves-dom-def
      preserves-cod-def preserves-id-def preserves-comp-def)
qed

```

end

4 HomFunctors

```

theory HomFunctors
imports SetCat Functors
begin

```

```

locale into-set = two-cats +
  assumes AA = (AA::('o,'a,'m)category-scheme)
  fixes U and Set
  defines U  $\equiv$  (UNIV::'a set)
  defines Set  $\equiv$  set-cat U
  assumes BB-Set: BB = Set
  fixes homf (Hom'(-,'-'))
  defines homf A  $\equiv$  ( $\lambda$ 
    om = ( $\lambda$ B $\in$ Ob. Hom A B),
    am = ( $\lambda$ f $\in$ Ar. ( $\lambda$ set-dom=Hom A (Dom f),set-func=( $\lambda$ g $\in$ Hom A (Dom f). f  $\cdot$ 
g),set-cod=Hom A (Cod f)))
  )

```

lemma (in into-set) homf-preserves-arrows:

```

  Hom(A,-)a : Ar  $\rightarrow$  ar Set
proof (rule funcsetI)
  fix f
  assume f: f  $\in$  Ar
  thus Hom(A,-)a f  $\in$  ar Set
proof (simp add: homf-def Set-def set-cat-def set-arrow-def U-def)
  have 1: (op  $\cdot$ ) : Hom (Dom f) (Cod f)  $\rightarrow$  Hom A (Dom f)  $\rightarrow$  Hom A (Cod f)
  ..
  have 2: f  $\in$  Hom (Dom f) (Cod f) using f by (simp add: hom-def)
  from 1 and 2 have 3: (op  $\cdot$ ) f : Hom A (Dom f)  $\rightarrow$  Hom A (Cod f)
  by (rule funcset-mem)
  show ( $\lambda$ g $\in$ Hom A (Dom f). f  $\cdot$  g) : Hom A (Dom f)  $\rightarrow$  Hom A (Cod f)
proof (rule funcsetI)
  fix g'
  assume g'  $\in$  Hom A (Dom f)
  from 3 and this show ( $\lambda$ g $\in$ Hom A (Dom f). f  $\cdot$  g) g'  $\in$  Hom A (Cod f)
  by simp (rule funcset-mem)
  qed
qed
qed

```

lemma (in into-set) homf-preserves-objects:

```

  Hom(A,-)o : Ob  $\rightarrow$  ob Set
proof (rule funcsetI)
  fix B
  assume B: B  $\in$  Ob
  have Hom(A,-)o B = Hom A B
  using B by (simp add: homf-def)
  moreover have ...  $\in$  ob Set
  by (simp add: U-def Set-def set-cat-def)
  ultimately show Hom(A,-)o B  $\in$  ob Set by simp
qed

```

lemma (in *into-set*) *homf-preserves-dom*:
 assumes $f: f \in Ar$
 shows $Hom(A,-)_o (Dom f) = dom Set (Hom(A,-)_a f)$
proof –
 have $Dom f \in Ob$ using f ..
 hence 1: $Hom(A,-)_o (Dom f) = Hom A (Dom f)$
 using f by (simp add: *homf-def*)
 have 2: $dom Set (Hom(A,-)_a f) = Hom A (Dom f)$
 using f by (simp add: *Set-def homf-def*)
 from 1 and 2 show *?thesis* by simp
qed

lemma (in *into-set*) *homf-preserves-cod*:
 assumes $f: f \in Ar$
 shows $Hom(A,-)_o (Cod f) = cod Set (Hom(A,-)_a f)$
proof –
 have $Cod f \in Ob$ using f ..
 hence 1: $Hom(A,-)_o (Cod f) = Hom A (Cod f)$
 using f by (simp add: *homf-def*)
 have 2: $cod Set (Hom(A,-)_a f) = Hom A (Cod f)$
 using f by (simp add: *Set-def homf-def*)
 from 1 and 2 show *?thesis* by simp
qed

lemma (in *into-set*) *homf-preserves-id*:
 assumes $B: B \in Ob$
 shows $Hom(A,-)_a (Id B) = id Set (Hom(A,-)_o B)$
proof –
 have 1: $Id B \in Ar$ using B ..
 have 2: $Dom (Id B) = B$
 using B by (rule *AA.id-dom-cod*)
 have 3: $Cod (Id B) = B$
 using B by (rule *AA.id-dom-cod*)
 have 4: $(\lambda g \in Hom A B. (Id B) \cdot g) = (\lambda g \in Hom A B. g)$
 by (rule *ext*) (auto simp add: *hom-def*)
 have $Hom(A,-)_a (Id B) = \{\}$
 $set-dom = Hom A B,$
 $set-func = (\lambda g \in Hom A B. g),$
 $set-cod = Hom A B\}$
 by (simp add: *homf-def 1 2 3 4*)
 also have $\dots = id Set (Hom(A,-)_o B)$
 using B by (simp add: *Set-def U-def set-cat-def set-id-def homf-def*)
 finally show *?thesis* .
qed

lemma (in *into-set*) *homf-preserves-comp*:

```

assumes  $f: f \in Ar$ 
and  $g: g \in Ar$ 
and  $fg: Cod f = Dom g$ 
shows  $Hom(A,-)_a (g \cdot f) = (Hom(A,-)_a g) \odot (Hom(A,-)_a f)$ 
proof –
have 1:  $g \cdot f \in Ar$  using assms ..
have 2:  $Dom (g \cdot f) = Dom f$  using f g fg ..
have 3:  $Cod (g \cdot f) = Cod g$  using f g fg ..
have lhs: Hom(A,-)_a (g \cdot f) =
  set-dom=Hom A (Dom f),
  set-func=( $\lambda h \in Hom A (Dom f). (g \cdot f) \cdot h$ ),
  set-cod=Hom A (Cod g)
by (simp add: homf-def 1 2 3)
have 4:  $set-dom ((Hom(A,-)_a g) \odot (Hom(A,-)_a f)) = Hom A (Dom f)$ 
using f by (simp add: set-comp-def homf-def)
have 5:  $set-cod ((Hom(A,-)_a g) \odot (Hom(A,-)_a f)) = Hom A (Cod g)$ 
using g by (simp add: set-comp-def homf-def)
have set-func ((Hom(A,-)_a g) \odot (Hom(A,-)_a f))
  = compose (Hom A (Dom f)) ( $\lambda y \in Hom A (Dom g). g \cdot y$ ) ( $\lambda x \in Hom A (Dom$ 
f).  $f \cdot x$ )
using f g by (simp add: set-comp-def homf-def)
also have  $\dots = (\lambda h \in Hom A (Dom f). (g \cdot f) \cdot h)$ 
proof (
  rule extensionalityI,
  rule compose-extensional,
  rule restrict-extensional,
  simp)
fix h
assume 10:  $h \in Hom A (Dom f)$ 
hence 11:  $f \cdot h \in Hom A (Dom g)$ 
proof –
from 10 have  $h \in Ar$  by (simp add: hom-def)
have 100:  $(op \cdot) : Hom (Dom f) (Dom g) \rightarrow Hom A (Dom f) \rightarrow Hom A$ 
(Dom g)
by (rule AA.comp-types)
have  $f \in Hom (Dom f) (Cod f)$  using f by (simp add: hom-def)
hence 101:  $f \in Hom (Dom f) (Dom g)$  using fg by simp
from 100 and 101
have  $(op \cdot) f : Hom A (Dom f) \rightarrow Hom A (Dom g)$ 
by (rule funcset-mem)
from this and 10
show  $f \cdot h \in Hom A (Dom g)$ 
by (rule funcset-mem)
qed
hence  $Cod (f \cdot h) = Dom g$ 
and  $Dom (f \cdot h) = A$ 
and  $f \cdot h \in Ar$ 
by (simp-all add: hom-def)
thus compose (Hom A (Dom f)) ( $\lambda y \in Hom A (Dom g). g \cdot y$ ) ( $\lambda x \in Hom A$ 

```

```

(Dom f). f · x) h =
  (g · f) · h
  using f g fg 10 by (simp add: compose-def 10 11 hom-def)
qed
finally have 6: set-func ((Hom(A,-)a g) ∘ (Hom(A,-)a f))
  = (λh∈Hom A (Dom f). (g · f) · h) .
from 4 and 5 and 6
have rhs: (Hom(A,-)a g) ∘ (Hom(A,-)a f) = (|
  set-dom=Hom A (Dom f),
  set-func=(λh∈Hom A (Dom f). (g · f) · h),
  set-cod=Hom A (Cod g)|)
by simp
show ?thesis
by (simp add: lhs rhs)
qed

```

theorem (in *into-set*) *homf-into-set*:

Functor Hom(A,-) : AA → Set

proof (*intro functor.intro functor-axioms.intro*)

show *Hom(A,-)_a : Ar → ar Set*

by (*rule homf-preserves-arrows*)

show *Hom(A,-)_o : Ob → ob Set*

by (*rule homf-preserves-objects*)

show $\forall f \in Ar. Hom(A,-) \circ (Dom f) = dom Set (Hom(A,-) \text{ a } f)$

by (*intro ballI*) (*rule homf-preserves-dom*)

show $\forall f \in Ar. Hom(A,-) \circ (Cod f) = cod Set (Hom(A,-) \text{ a } f)$

by (*intro ballI*) (*rule homf-preserves-cod*)

show $\forall B \in Ob. Hom(A,-) \text{ a } (Id B) = id Set (Hom(A,-) \text{ o } B)$

by (*intro ballI*) (*rule homf-preserves-id*)

show $\forall f \in Ar. \forall g \in Ar.$

Cod f = Dom g →

Hom(A,-)_a (g · f) = comp Set (Hom(A,-)_a g) (Hom(A,-)_a f)

by (*intro ballI impI, simp add: Set-def set-cat-def*) (*rule homf-preserves-comp*)

show *two-cats AA Set*

proof *intro-locales*

show *category Set*

by (*unfold Set-def, rule set-cat-cat*)

show *two-cats-axioms AA Set*

proof **qed** *rule+*

qed

qed

end

5 Natural Transformations

theory *NatTrans*

imports *Functors*
begin

locale *natural-transformation* = *two-cats* +
fixes *F* **and** *G* **and** *u*
assumes *Functor* *F* : *AA* \longrightarrow *BB*
and *Functor* *G* : *AA* \longrightarrow *BB*
and *u* : *ob AA* \rightarrow *ar BB*
and *u* \in *extensional (ob AA)*
and $\forall A \in \text{Ob. } u A \in \text{Hom}_2 (F_o A) (G_o A)$
and $\forall A \in \text{Ob. } \forall B \in \text{Ob. } \forall f \in \text{Hom } A B. (G_a f) \cdot_2 (u A) = (u B) \cdot_2 (F_a f)$

abbreviation

nt-syn ($- : - \Rightarrow -$ in *Func* '($- , -$) [*81*]) **where**
 $u : F \Rightarrow G$ in *Func*(*AA*, *BB*) \equiv *natural-transformation AA BB F G u*

locale *endoNT* = *natural-transformation* + *one-cat*

theorem (**in** *endoNT*) *id-restrict-natural*:

$(\lambda A \in \text{Ob. } \text{Id } A) : (\text{id-func } AA) \Rightarrow (\text{id-func } AA)$ in *Func*(*AA*, *AA*)

proof (*intro natural-transformation.intro natural-transformation-axioms.intro*

two-cats.intro two-cats-axioms.intro ballI)

show $(\lambda A \in \text{Ob. } \text{Id } A) : \text{Ob} \rightarrow \text{Ar}$

by (*rule funcsetI*) *auto*

show $(\lambda A \in \text{Ob. } \text{Id } A) \in \text{extensional } (\text{Ob})$

by (*rule restrict-extensional*)

fix *A*

assume *A*: $A \in \text{Ob}$

hence $\text{Id } A \in \text{Hom } A A ..$

thus $(\lambda X \in \text{Ob. } \text{Id } X) A \in \text{Hom } ((\text{id-func } AA)_o A) ((\text{id-func } AA)_o A)$

using *A* **by** (*simp add: id-func-def*)

fix *B* **and** *f*

assume *B*: $B \in \text{Ob}$

and $f \in \text{Hom } A B$

hence $f \in \text{Ar}$ **and** $A = \text{Dom } f$ **and** $B = \text{Cod } f$ **and** $\text{Dom } f \in \text{Ob}$ **and** $\text{Cod } f \in \text{Ob}$

using *A* **by** (*simp-all add: hom-def*)

thus $(\text{id-func } AA)_a f \cdot (\lambda A \in \text{Ob. } \text{Id } A) A$

$= (\lambda A \in \text{Ob. } \text{Id } A) B \cdot (\text{id-func } AA)_a f$

by (*simp add: id-func-def*)

qed (*auto intro: id-func-functor, unfold-locales, unfold-locales*)

end

6 Yoneda Lemma

```
theory Yoneda
imports HomFunctors NatTrans
begin
```

6.1 The Sandwich Natural Transformation

```
locale Yoneda = functor + into-set +
  assumes AA = (AA::('o,'a,'m)category-scheme)
  fixes sandwich :: ['o,'a,'o]  $\Rightarrow$  'a set-arrow ( $\sigma'(-,-)$ )
  defines sandwich A a  $\equiv$  ( $\lambda B \in Ob.$  (
    set-dom = Hom A B,
    set-func = ( $\lambda f \in Hom A B.$  set-func (Fa f) a),
    set-cod = Fo B
  ))
  fixes unsandwich :: ['o,'o]  $\Rightarrow$  'a set-arrow  $\Rightarrow$  'a ( $\sigma^{-}'(-,-)$ )
  defines unsandwich A u  $\equiv$  set-func (u A) (Id A)
```

lemma (in Yoneda) *F-into-set*:

Functor F : AA \longrightarrow Set

proof –

from *F-axioms* **have** *Functor F : AA \longrightarrow BB* **by** *intro-locales*

thus *?thesis*

by (*simp only: BB-Set*)

qed

lemma (in Yoneda) *F-comp-func*:

assumes *1: A \in Ob* **and** *2: B \in Ob* **and** *3: C \in Ob*

and *4: g \in Hom A B* **and** *5: f \in Hom B C*

shows *set-func (F_a (f \cdot g)) = compose (F_o A) (set-func (F_a f)) (set-func (F_a g))*

proof –

from *4* **and** *5*

have *7: Cod g = Dom f*

and *8: g \in Ar*

and *9: f \in Ar*

and *10: Dom g = A*

by (*simp-all add: hom-def*)

from *F-preserves-dom* **and** *8* **and** *10*

have *11: set-dom (F_a g) = F_o A*

by (*simp add: preserves-dom-def BB-Set Set-def*) *auto*

from *F-preserves-comp* **and** *7* **and** *8* **and** *9*

have *F_a (f \cdot g) = (F_a f) \cdot_2 (F_a g)*

by (*simp add: preserves-comp-def*)

hence *set-func (F_a (f \cdot g)) = set-func ((F_a f) \odot (F_a g))*

by (*simp add: BB-Set Set-def*)

also have $\dots =$ *compose (F_o A) (set-func (F_a f)) (set-func (F_a g))*

by (*simp add: set-comp-def 11*)

finally show *?thesis* .
 qed

lemma (in *Yoneda*) *sandwich-funcset*:

assumes $A: A \in Ob$
 and $a \in F_o A$
 shows $\sigma(A,a) : Ob \rightarrow ar Set$
 proof (rule *funcsetI*)
 fix B
 assume $B: B \in Ob$
 thus $\sigma(A,a) B \in ar Set$
 proof (simp add: *Set-def sandwich-def set-cat-def*)
 show *set-arrow* $U \{$
 set-dom = $Hom A B$,
 set-func = $\lambda f \in Hom A B. set-func (F_a f) a$,
 set-cod = $F_o B$)
 proof (simp add: *set-arrow-def, intro conjI*)
 show $Hom A B \subseteq U$ and $F_o B \subseteq U$
 by (simp-all add: *U-def*)
 show $(\lambda f \in Hom A B. set-func (F_a f) a) \in Hom A B \rightarrow F_o B$
 proof (rule *funcsetI, simp*)
 fix f
 assume $f: f \in Hom A B$
 with $A B$ have $F_a f \in Hom_2 (F_o A) (F_o B)$
 by (rule *functors-preserve-homsets*)
 hence $F_a f \in ar Set$
 and *set-dom* $(F_a f) = (F_o A)$
 and *set-cod* $(F_a f) = (F_o B)$
 by (simp-all add: *hom-def BB-Set Set-def*)
 hence *set-func* $(F_a f) : (F_o A) \rightarrow (F_o B)$
 by (simp add: *Set-def set-cat-def set-arrow-def*)
 thus *set-func* $(F_a f) a \in F_o B$
 using $\langle a \in F_o A \rangle$
 by (rule *funcset-mem*)
 qed
 qed
 qed
 qed

lemma (in *Yoneda*) *sandwich-type*:

assumes $A: A \in Ob$ and $B: B \in Ob$
 and $a \in F_o A$
 shows $\sigma(A,a) B \in hom Set (Hom A B) (F_o B)$
 proof –
 have $\sigma(A,a) \in Ob \rightarrow Ar Set$
 using A and $\langle a \in F_o A \rangle$ by (rule *sandwich-funcset*)
 hence $\sigma(A,a) B \in ar Set$
 using B by (rule *funcset-mem*)

thus *?thesis*
 using B by (*simp add: sandwich-def hom-def Set-def*)
 qed

lemma (in *Yoneda*) *sandwich-commutes*:

assumes $AOb: A \in Ob$ and $BOb: B \in Ob$ and $COb: C \in Ob$
 and $aFa: a \in F_o A$
 and $fBC: f \in Hom B C$
 shows $(F_a f) \odot (\sigma(A,a) B) = (\sigma(A,a) C) \odot (Hom(A,-)_a f)$

proof –

from fBC have 1: $f \in Ar$ and 2: $Dom f = B$ and 3: $Cod f = C$
 by (*simp-all add: hom-def*)

from BOb have *set-dom* $((F_a f) \odot (\sigma(A,a) B)) = Hom A B$
 by (*simp add: set-comp-def sandwich-def*)

also have $\dots = set-dom ((\sigma(A,a) C) \odot (Hom(A,-)_a f))$
 by (*simp add: set-comp-def homf-def 1 2*)

finally have *set-dom-eq*:

$set-dom ((F_a f) \odot (\sigma(A,a) B))$
 $= set-dom ((\sigma(A,a) C) \odot (Hom(A,-)_a f)) .$

from $BOb COb fBC$ have $(F_a f) \in Hom_2 (F_o B) (F_o C)$
 by (*rule functors-preserve-homsets*)

hence *set-cod* $((F_a f) \odot (\sigma(A,a) B)) = F_o C$

by (*simp add: set-comp-def BB-Set Set-def set-cat-def hom-def*)

also from COb

have $\dots = set-cod ((\sigma(A,a) C) \odot (Hom(A,-)_a f))$

by (*simp add: set-comp-def sandwich-def*)

finally have *set-cod-eq*:

$set-cod ((F_a f) \odot (\sigma(A,a) B))$
 $= set-cod ((\sigma(A,a) C) \odot (Hom(A,-)_a f)) .$

from AOb and BOb and COb and fBC and aFa

have *set-func-lhs*:

$set-func ((F_a f) \odot (\sigma(A,a) B)) =$
 $(\lambda g \in Hom A B. set-func (F_a (f \cdot g)) a)$

apply (*simp add: set-comp-def sandwich-def compose-def*)

apply (*rule extensionalityI, rule restrict-extensional, rule restrict-extensional*)

by (*simp add: F-comp-func compose-def*)

have $(op \cdot) : Hom B C \rightarrow Hom A B \rightarrow Hom A C ..$

from *this* and fBC

have *opfType*: $(op \cdot) f : Hom A B \rightarrow Hom A C$

by (*rule funcset-mem*)

from 1 and 2

have *set-func* $((\sigma(A,a) C) \odot (Hom(A,-)_a f)) =$

$(\lambda g \in Hom A B. set-func (\sigma(A,a) C) (f \cdot g))$

apply (*simp add: set-comp-def homf-def*)

apply (*simp add: compose-def*)

apply (*rule extensionalityI, rule restrict-extensional, rule restrict-extensional*)

by *auto*

also from COb and *opfType*

```

have ... = ( $\lambda g \in \text{Hom } A \ B. \text{ set-func } (F \text{ }_a (f \cdot g)) \ a$ )
  apply (simp add: sandwich-def)
  apply (rule extensionalityI, rule restrict-extensional, rule restrict-extensional)
  by (simp add: funcset-mem)
finally have set-func-rhs:
  set-func (( $\sigma(A, a) \ C$ )  $\odot$  (Hom( $A, -$ )  $_a f$ )) =
  ( $\lambda g \in \text{Hom } A \ B. \text{ set-func } (F \text{ }_a (f \cdot g)) \ a$ ) .
from set-func-lhs and set-func-rhs have
  set-func (( $F \text{ }_a f$ )  $\odot$  ( $\sigma(A, a) \ B$ ))
  = set-func (( $\sigma(A, a) \ C$ )  $\odot$  (Hom( $A, -$ )  $_a f$ ))
  by simp
with set-dom-eq and set-cod-eq show ?thesis
  by simp
qed

```

lemma (in *Yoneda*) *sandwich-natural*:

```

assumes  $A \in \text{Ob}$ 
and  $a \in F \circ A$ 
shows  $\sigma(A, a) : \text{Hom}(A, -) \Rightarrow F$  in Func( $AA, \text{Set}$ )
proof (intro natural-transformation.intro natural-transformation-axioms.intro two-cats.intro)
show category  $AA$  ..
show category  $\text{Set}$ 
  by (simp only: Set-def)(rule set-cat-cat)
show Functor Hom( $A, -$ ) :  $AA \longrightarrow \text{Set}$ 
  by (rule homf-into-set)
show Functor  $F : AA \longrightarrow \text{Set}$ 
  by (rule F-into-set)
show  $\forall B \in \text{Ob}. \sigma(A, a) \ B \in \text{hom } \text{Set} \ (\text{Hom}(A, -) \circ B) \ (F \circ B)$ 
  using assms by (auto simp add: homf-def intro: sandwich-type)
show  $\sigma(A, a) : \text{Ob} \rightarrow \text{ar } \text{Set}$ 
  using assms by (rule sandwich-funcset)
show  $\sigma(A, a) \in \text{extensional } (\text{Ob})$ 
  unfolding sandwich-def by (rule restrict-extensional)
show  $\forall B \in \text{Ob}. \forall C \in \text{Ob}. \forall f \in \text{Hom } B \ C.$ 
  comp  $\text{Set} \ (F \text{ }_a f) \ (\sigma(A, a) \ B) = \text{comp } \text{Set} \ (\sigma(A, a) \ C) \ (\text{Hom}(A, -) \text{ }_a f)$ 
  using assms by (auto simp add: Set-def intro: sandwich-commutes)
qed (intro two-cats-axioms.intro, simp-all)

```

6.2 Sandwich Components are Bijective

lemma (in *Yoneda*) *unsandwich-left-inverse*:

```

assumes  $1: A \in \text{Ob}$ 
and  $2: a \in F \circ A$ 
shows  $\sigma^\leftarrow(A, \sigma(A, a)) = a$ 
proof -
from  $1$  have  $\text{Id } A \in \text{Hom } A \ A$  ..
with  $1$ 
have  $3: \sigma^\leftarrow(A, \sigma(A, a)) = \text{set-func } (F \text{ }_a (\text{Id } A)) \ a$ 

```

by (*simp add: sandwich-def homf-def unsandwich-def*)
from *F-preserves-id and 1*
have $4: F_a (Id A) = id Set (F_o A)$
 by (*simp add: preserves-id-def BB-Set*)
from *F-preserves-objects and 1*
have $F_o A \in Ob_2$
 by (*rule funcset-mem*)
hence $F_o A \subseteq U$
 by (*simp add: BB-Set Set-def set-cat-def*)
with 2
have $5: set-func (id Set (F_o A)) a = a$
 by (*simp add: Set-def set-id-def*)
show *?thesis*
 by (*simp add: 3 4 5*)
qed

lemma (in *Yoneda*) *unsandwich-right-inverse:*

assumes $1: A \in Ob$
and $2: u : Hom(A, -) \Rightarrow F$ in *Func(AA, Set)*
shows $\sigma(A, \sigma^{\leftarrow}(A, u)) = u$
proof (*rule extensionalityI*)
show $\sigma(A, \sigma^{\leftarrow}(A, u)) \in extensional (Ob)$
 by (*unfold sandwich-def, rule restrict-extensional*)
from 2 **show** $u \in extensional (Ob)$
 by (*simp add: natural-transformation-def natural-transformation-axioms-def*)
fix B
assume $3: B \in Ob$
with 1
have *one:* $\sigma(A, \sigma^{\leftarrow}(A, u)) B = ()$
 $set-dom = Hom A B,$
 $set-func = (\lambda f \in Hom A B. (set-func (F_a f)) (set-func (u A) (Id A))),$
 $set-cod = F_o B ()$
 by (*simp add: sandwich-def unsandwich-def*)
from 1 **have** $Hom(A, -)_o A = Hom A A$
 by (*simp add: homf-def*)
with 1 **and** 2 **have** $(u A) \in hom Set (Hom A A) (F_o A)$
 by (*simp add: natural-transformation-def natural-transformation-axioms-def, auto*)
hence $set-dom (u A) = Hom A A$
 by (*simp add: hom-def Set-def*)
with 1 **have** *applicable:* $Id A \in set-dom (u A)$
 by (*simp*)(*rule*)
have *two:* $(\lambda f \in Hom A B. (set-func (F_a f)) (set-func (u A) (Id A)))$
 $= (\lambda f \in Hom A B. (set-func ((F_a f) \odot (u A)) (Id A)))$
 by (*rule extensionalityI,*
 rule restrict-extensional, rule restrict-extensional,
 simp add: set-comp-def compose-def applicable)
from 2

have $(\forall X \in Ob. \forall Y \in Ob. \forall f \in Hom\ X\ Y. (F_a\ f) \cdot_2 (u\ X) = (u\ Y) \cdot_2 (Hom(A, -)_a\ f))$
by (*simp add: natural-transformation-def natural-transformation-axioms-def BB-Set*)
with 1 and 3
have *three*: $(\lambda f \in Hom\ A\ B. (set-func\ ((F_a\ f) \odot (u\ A))\ (Id\ A)))$
 $= (\lambda f \in Hom\ A\ B. (set-func\ ((u\ B) \odot (Hom(A, -)_a\ f))\ (Id\ A)))$
apply (*simp add: BB-Set Set-def*)
apply (*rule extensionalityI*)
apply (*rule restrict-extensional, rule restrict-extensional*)
by *simp*
have $\forall f \in Hom\ A\ B. set-dom\ (Hom(A, -)_a\ f) = Hom\ A\ A$
by (*intro ballI, simp add: homf-def hom-def*)
have *roolz*: $\bigwedge f. f \in Hom\ A\ B \implies set-dom\ (Hom(A, -)_a\ f) = Hom\ A\ A$
by (*simp add: homf-def hom-def*)
from 1 **have** *rooly*: $Id\ A \in Hom\ A\ A$..
have *roolx*: $\bigwedge f. f \in Hom\ A\ B \implies f \in Ar$
by (*simp add: hom-def*)
have *roolw*: $\bigwedge f. f \in Hom\ A\ B \implies Id\ A \in Hom\ A\ (Dom\ f)$
proof-
fix *f*
assume $f \in Hom\ A\ B$
hence $Dom\ f = A$ **by** (*simp add: hom-def*)
thus $Id\ A \in Hom\ A\ (Dom\ f)$
by (*simp add: rooly*)
qed
have *annoying*: $\bigwedge f. f \in Hom\ A\ B \implies Id\ A = Id\ (Dom\ f)$
by (*simp add: hom-def*)
have $(\lambda f \in Hom\ A\ B. (set-func\ ((u\ B) \odot (Hom(A, -)_a\ f))\ (Id\ A)))$
 $= (\lambda f \in Hom\ A\ B. (compose\ (Hom\ A\ A)\ (set-func\ (u\ B))\ (set-func\ (Hom(A, -)_a\ f))))\ (Id\ A))$
apply (*rule extensionalityI*)
apply (*rule restrict-extensional, rule restrict-extensional*)
by (*simp add: compose-def set-comp-def roolz rooly*)
also have ... $= (\lambda f \in Hom\ A\ B. (set-func\ (u\ B)\ f))$
apply (*rule extensionalityI*)
apply (*rule restrict-extensional, rule restrict-extensional*)
apply (*simp add: compose-def homf-def rooly roolx roolw*)
apply (*simp only: annoying*)
apply (*simp add: roolx id-right*)
done
finally have *four*:
 $(\lambda f \in Hom\ A\ B. (set-func\ ((u\ B) \odot (Hom(A, -)_a\ f))\ (Id\ A)))$
 $= (\lambda f \in Hom\ A\ B. (set-func\ (u\ B)\ f))$.
from 2 and 3
have *uBhom*: $u\ B \in hom\ Set\ (Hom(A, -)_o\ B)\ (F_o\ B)$
by (*simp add: natural-transformation-def natural-transformation-axioms-def*)
with 3
have *five*: $set-dom\ (u\ B) = Hom\ A\ B$

```

  by (simp add: hom-def homf-def Set-def set-cat-def)
from uBhom
have six: set-cod (u B) = Fo B
  by (simp add: hom-def homf-def Set-def set-cat-def)
have seven: restrict (set-func (u B)) (Hom A B) = set-func (u B)
  apply (rule extensionalityI)
  apply (rule restrict-extensional)
proof-
  from uBhom have u B ∈ ar Set
    by (simp add: hom-def)
  hence almost: set-func (u B) ∈ extensional (set-dom (u B))
    by (simp add: Set-def set-cat-def set-arrow-def)
  from almost and five
  show set-func (u B) ∈ extensional (Hom A B)
    by simp
  fix f
  assume f ∈ Hom A B
  thus restrict (set-func (u B)) (Hom A B) f = set-func (u B) f
    by simp
qed
from one and two and three and four and five and six and seven
show σ(A,σ←(A,u)) B = u B
  by simp
qed

```

In order to state the lemma, we must rectify a curious omission from the Isabelle/HOL library. They define the idea of injectivity on a given set, but surjectivity is only defined relative to the entire universe of the target type.

definition

```

surj-on :: ['a ⇒ 'b, 'a set, 'b set] ⇒ bool where
surj-on f A B ⟷ (∀ y∈B. ∃ x∈A. f(x)=y)

```

definition

```

bij-on :: ['a ⇒ 'b, 'a set, 'b set] ⇒ bool where
bij-on f A B ⟷ inj-on f A & surj-on f A B

```

definition

```

equinumerous :: ['a set, 'b set] ⇒ bool (infix ≅ 40) where
equinumerous A B ⟷ (∃ f. bij-on f A B)

```

theorem (in Yoneda) Yoneda:

```

  assumes 1: A ∈ Ob
  shows Fo A ≅ {u. u : Hom(A,-) ⇒ F in Func(AA,Set)}
  apply (unfold equinumerous-def bij-on-def surj-on-def inj-on-def)
  apply (intro exI conjI bexI ballI impI)
proof-
  — Sandwich is injective
  fix x and y
  assume 2: x ∈ Fo A and 3: y ∈ Fo A

```

```

and 4:  $\sigma(A,x) = \sigma(A,y)$ 
hence  $\sigma^{\leftarrow}(A,\sigma(A,x)) = \sigma^{\leftarrow}(A,\sigma(A,y))$ 
  by simp
with unsandwich-left-inverse
show  $x = y$ 
  by (simp add: 1 2 3)
next
  — Sandwich covers F A
  fix  $u$ 
  assume  $u \in \{y. y : \text{Hom}(A,-) \Rightarrow F \text{ in } \text{Func}(AA,\text{Set})\}$ 
  hence 2:  $u : \text{Hom}(A,-) \Rightarrow F \text{ in } \text{Func}(AA,\text{Set})$ 
    by simp
  with 1 show  $\sigma(A,\sigma^{\leftarrow}(A,u)) = u$ 
    by (rule unsandwich-right-inverse)
  — Sandwich is into F A
  from 1 and 2
  have  $u A \in \text{hom Set}(\text{Hom } A A)(F_{\circ} A)$ 
    by (simp add: natural-transformation-def natural-transformation-axioms-def
homf-def)
  hence  $u A \in \text{ar Set}$  and  $\text{dom Set}(u A) = \text{Hom } A A$  and  $\text{cod Set}(u A) = F_{\circ} A$ 
    by (simp-all add: hom-def)
  hence  $uA\text{funcset}: \text{set-func}(u A) : (\text{Hom } A A) \rightarrow (F_{\circ} A)$ 
    by (simp add: Set-def set-cat-def set-arrow-def)
  from 1 have  $\text{Id } A \in \text{Hom } A A$  ..
  with  $uA\text{funcset}$ 
  show  $\sigma^{\leftarrow}(A,u) \in F_{\circ} A$ 
    by (simp add: unsandwich-def, rule funcset-mem)
qed

end

```

References

- [O’K04] Greg O’Keefe. Towards a readable formalisation of category theory. In Mike Atkinson, editor, *Computing: The Australasian Theory Symposium*, volume 91 of *Electronic Notes in Theoretical Computer Science*, pages 212–228. Elsevier, 2004.