

Category Theory to Yoneda's Lemma

Greg O'Keefe

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This development proves Yoneda's lemma and aims to be readable by humans. It only defines what is needed for the lemma: categories, functors and natural transformations. Limits, adjunctions and other important concepts are not included.

There is no explanation or discussion in this document. See [O'K04] for this and a survey of category theory formalisations.

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1 Categories

```
theory Cat
imports FuncSet
begin
```

1.1 Definitions

```
record ('o, 'a) category =
  ob :: 'o set (Ob1 70)
  ar :: 'a set (Ar1 70)
  dom :: 'a ⇒ 'o (Dom1 - [81] 70)
  cod :: 'a ⇒ 'o (Cod1 - [81] 70)
  id :: 'o ⇒ 'a (Id1 - [81] 80)
  comp :: 'a ⇒ 'a ⇒ 'a (infixl · 60)
```

definition

```
hom :: [('o, 'a, 'm) category-scheme, 'o, 'o] ⇒ 'a set (Hom1 - -) where
hom CC A B = { f. f ∈ ar CC & dom CC f = A & cod CC f = B }
```

```
locale category =
```

```
  fixes CC (structure)
```

```
  assumes dom-object [intro]:
```

```
    f ∈ Ar ⇒ Dom f ∈ Ob
```

```
  and cod-object [intro]:
```

```
    f ∈ Ar ⇒ Cod f ∈ Ob
```

```
  and id-left [simp]:
```

```
    f ∈ Ar ⇒ Id (Cod f) · f = f
```

```
  and id-right [simp]:
```

```
    f ∈ Ar ⇒ f · Id (Dom f) = f
```

```
  and id-hom [intro]:
```

```
    A ∈ Ob ⇒ Id A ∈ Hom A A
```

```
  and comp-types [intro]:
```

```
     $\bigwedge A B C. (comp\ CC) : (Hom\ B\ C) \rightarrow (Hom\ A\ B) \rightarrow (Hom\ A\ C)$ 
```

```
  and comp-associative [simp]:
```

```
    f ∈ Ar ⇒ g ∈ Ar ⇒ h ∈ Ar
```

```
    ⇒ Cod h = Dom g ⇒ Cod g = Dom f
```

```
    ⇒ f · (g · h) = (f · g) · h
```

1.2 Lemmas

```
lemma (in category) homI:
```

```
  assumes f ∈ Ar and Dom f = A and Cod f = B
```

```
  shows f ∈ Hom A B
```

```
  ⟨proof⟩
```

```
lemma (in category) homE:
```

```
  assumes A ∈ Ob and B ∈ Ob and f ∈ Hom A B
```

```
  shows Dom f = A and Cod f = B
```

```
  ⟨proof⟩
```

```

lemma (in category) id-arrow [intro]:
  assumes  $A \in Ob$ 
  shows  $Id\ A \in Ar$ 
   $\langle proof \rangle$ 

lemma (in category) id-dom-cod:
  assumes  $A \in Ob$ 
  shows  $Dom\ (Id\ A) = A$  and  $Cod\ (Id\ A) = A$ 
   $\langle proof \rangle$ 

lemma (in category) compI [intro]:
  assumes  $f: f \in Ar$  and  $g: g \in Ar$  and  $Cod\ f = Dom\ g$ 
  shows  $g \cdot f \in Ar$ 
  and  $Dom\ (g \cdot f) = Dom\ f$ 
  and  $Cod\ (g \cdot f) = Cod\ g$ 
   $\langle proof \rangle$ 

end

```

2 Set is a Category

```

theory SetCat
imports Cat
begin

```

2.1 Definitions

```

record 'c set-arrow =
  set-dom :: 'c set
  set-func :: 'c  $\Rightarrow$  'c
  set-cod :: 'c set

```

definition

```

set-arrow :: ['c set, 'c set-arrow]  $\Rightarrow$  bool where
set-arrow  $U\ f \longleftrightarrow$   $set-dom\ f \subseteq U$  &  $set-cod\ f \subseteq U$ 
  &  $(set-func\ f): (set-dom\ f) \rightarrow (set-cod\ f)$ 
  &  $set-func\ f \in extensional\ (set-dom\ f)$ 

```

definition

```

set-id :: ['c set, 'c set]  $\Rightarrow$  'c set-arrow where
set-id  $U = (\lambda s \in Pow\ U. (\setminus set-dom=s, set-func=\lambda x \in s. x, set-cod=s))$ 

```

definition

```

set-comp :: ['c set-arrow, 'c set-arrow]  $\Rightarrow$  'c set-arrow (infix  $\odot$  70) where
set-comp  $g\ f =$ 
  ( $\setminus$ 

```

$set-dom = set-dom f,$
 $set-func = compose (set-dom f) (set-func g) (set-func f),$
 $set-cod = set-cod g$
 \rangle

definition

$set-cat :: 'c set \Rightarrow ('c set, 'c set-arrow) category$ **where**
 $set-cat U =$
 \langle
 $ob = Pow U,$
 $ar = \{f. set-arrow U f\},$
 $dom = set-dom,$
 $cod = set-cod,$
 $id = set-id U,$
 $comp = set-comp$
 \rangle

2.2 Simple Rules and Lemmas

lemma *set-objectI* [*intro*]: $A \subseteq U \Longrightarrow A \in ob (set-cat U)$
 $\langle proof \rangle$

lemma *set-objectE* [*intro*]: $A \in ob (set-cat U) \Longrightarrow A \subseteq U$
 $\langle proof \rangle$

lemma *set-homI* [*intro*]:
assumes $A \subseteq U$
and $B \subseteq U$
and $f : A \rightarrow B$
and $f \in extensional A$
shows $(\setminus set-dom=A, set-func=f, set-cod=B) \in hom (set-cat U) A B$
 $\langle proof \rangle$

lemma *set-dom* [*simp*]: $dom (set-cat U) f = set-dom f$
 $\langle proof \rangle$

lemma *set-cod* [*simp*]: $cod (set-cat U) f = set-cod f$
 $\langle proof \rangle$

lemma *set-id* [*simp*]: $id (set-cat U) A = set-id U A$
 $\langle proof \rangle$

lemma *set-comp* [*simp*]: $comp (set-cat U) g f = g \odot f$
 $\langle proof \rangle$

lemma *set-dom-cod-object-subset* [*intro*]:
assumes $f: f \in ar (set-cat U)$
shows $dom (set-cat U) f \in ob (set-cat U)$

and $\text{cod } (\text{set-cat } U) f \in \text{ob } (\text{set-cat } U)$
and $\text{set-cod } f \subseteq U$
and $\text{set-dom } f \subseteq U$
 $\langle \text{proof} \rangle$

In this context, $f \in \text{hom } A B$ is quite a strong claim.

lemma *set-homE* [*intro*]:
assumes $f: f \in \text{hom } (\text{set-cat } U) A B$
shows $A \subseteq U$
and $B \subseteq U$
and $\text{set-dom } f = A$
and $\text{set-func } f : A \rightarrow B$
and $\text{set-cod } f = B$
 $\langle \text{proof} \rangle$

2.3 Set is a Category

lemma *set-id-left*:
assumes $f: f \in \text{ar } (\text{set-cat } U)$
shows $\text{set-id } U (\text{set-cod } f) \odot f = f$
 $\langle \text{proof} \rangle$

lemma *set-id-right*:
assumes $f: f \in \text{ar } (\text{set-cat } U)$
shows $f \odot (\text{set-id } U (\text{set-dom } f)) = f$
 $\langle \text{proof} \rangle$

lemma *set-id-hom*:
assumes $A \in \text{ob } (\text{set-cat } U)$
shows $\text{id } (\text{set-cat } U) A \in \text{hom } (\text{set-cat } U) A A$
 $\langle \text{proof} \rangle$

lemma *set-comp-types*:
 $\text{comp } (\text{set-cat } U) \in \text{hom } (\text{set-cat } U) B C \rightarrow \text{hom } (\text{set-cat } U) A B \rightarrow \text{hom } (\text{set-cat } U) A C$
 $\langle \text{proof} \rangle$

We reason explicitly about the function component of the composite arrow, leaving the rest to the simplifier.

lemma *set-comp-associative*:
fixes f **and** g **and** h
assumes $f: f \in \text{ar } (\text{set-cat } U)$
and $g: g \in \text{ar } (\text{set-cat } U)$
and $h: h \in \text{ar } (\text{set-cat } U)$
and $hg: \text{cod } (\text{set-cat } U) h = \text{dom } (\text{set-cat } U) g$
and $gf: \text{cod } (\text{set-cat } U) g = \text{dom } (\text{set-cat } U) f$
shows $\text{comp } (\text{set-cat } U) f (\text{comp } (\text{set-cat } U) g h) = \text{comp } (\text{set-cat } U) (\text{comp } (\text{set-cat } U) f g) h$

<proof>

theorem *set-cat-cat*: *category (set-cat U)*

<proof>

end

3 Functors

theory *Functors*

imports *Cat*

begin

3.1 Definitions

record (*'o1, 'a1, 'o2, 'a2*) *functor* =
 om :: *'o1* \Rightarrow *'o2*
 am :: *'a1* \Rightarrow *'a2*

abbreviation

om-syn (*-* _o [81]) **where**
 $F_o \equiv om\ F$

abbreviation

am-syn (*-* _a [81]) **where**
 $F_a \equiv am\ F$

locale *two-cats* = *AA: category AA + BB: category BB*

for *AA (structure) and BB (structure) +*

assumes *AA = (AA :: ('o1, 'a1, 'm1)category-scheme)*

assumes *BB = (BB :: ('o2, 'a2, 'm2)category-scheme)*

fixes *preserves-dom* :: (*'o1, 'a1, 'o2, 'a2*)*functor* \Rightarrow *bool*

and *preserves-cod* :: (*'o1, 'a1, 'o2, 'a2*)*functor* \Rightarrow *bool*

and *preserves-id* :: (*'o1, 'a1, 'o2, 'a2*)*functor* \Rightarrow *bool*

and *preserves-comp* :: (*'o1, 'a1, 'o2, 'a2*)*functor* \Rightarrow *bool*

defines *preserves-dom* *G* \equiv

$\forall f \in Ar_1. G_o (Dom_1 f) = Dom_2 (G_a f)$

and *preserves-cod* *G* \equiv

$\forall f \in Ar_1. G_o (Cod_1 f) = Cod_2 (G_a f)$

and *preserves-id* *G* \equiv

$\forall A \in Ob_1. G_a (Id_1 A) = Id_2 (G_o A)$

and *preserves-comp* *G* \equiv

$\forall f \in Ar_1. \forall g \in Ar_1. Cod_1 f = Dom_1 g \longrightarrow G_a (g \cdot_1 f) = (G_a g) \cdot_2 (G_a f)$

locale *functor* = *two-cats +*

fixes *F (structure)*

assumes *F-preserves-arrows*: $F_a : Ar_1 \rightarrow Ar_2$

and F -preserves-objects: $F_o : Ob_1 \rightarrow Ob_2$
and F -preserves-dom: preserves-dom F
and F -preserves-cod: preserves-cod F
and F -preserves-id: preserves-id F
and F -preserves-comp: preserves-comp F
notes F -axioms = F -preserves-arrows F -preserves-objects F -preserves-dom
 F -preserves-cod F -preserves-id F -preserves-comp
notes $func$ -pred-defs = preserves-dom-def preserves-cod-def preserves-id-def preserves-comp-def

This gives us nicer notation for asserting that things are functors.

abbreviation

Functor (*Functor* - : - \longrightarrow - [81]) **where**
Functor $F : AA \longrightarrow BB \equiv functor\ AA\ BB\ F$

3.2 Simple Lemmas

For example:

lemma (**in** *functor*) *Functor* $F : AA \longrightarrow BB$ $\langle proof \rangle$

lemma *functors-preserve-arrows* [*intro*]:

assumes *Functor* $F : AA \longrightarrow BB$

and $f \in ar\ AA$

shows $F_a\ f \in ar\ BB$

$\langle proof \rangle$

lemma (**in** *functor*) *functors-preserve-homsets*:

assumes 1: $A \in Ob_1$

and 2: $B \in Ob_1$

and 3: $f \in Hom_1\ A\ B$

shows $F_a\ f \in Hom_2\ (F_o\ A)\ (F_o\ B)$

$\langle proof \rangle$

lemma *functors-preserve-objects* [*intro*]:

assumes *Functor* $F : AA \longrightarrow BB$

and $A \in ob\ AA$

shows $F_o\ A \in ob\ BB$

$\langle proof \rangle$

3.3 Identity Functor

definition

id -func :: ('o,'a,'m) *category-scheme* \Rightarrow ('o,'a,'o,'a) *functor* **where**
 id -func $CC = (\!|om=(\lambda A \in ob\ CC. A), am=(\lambda f \in ar\ CC. f)\!|)$

locale *one-cat* = *two-cats* +

assumes *endo*: $BB = AA$

lemma (in *one-cat*) *id-func-preserves-arrows*:
 shows $(id\text{-func } AA)_a : Ar \rightarrow Ar$
 ⟨*proof*⟩

lemma (in *one-cat*) *id-func-preserves-objects*:
 shows $(id\text{-func } AA)_o : Ob \rightarrow Ob$
 ⟨*proof*⟩

lemma (in *one-cat*) *id-func-preserves-dom*:
 shows *preserves-dom* (*id-func* *AA*)
 ⟨*proof*⟩

lemma (in *one-cat*) *id-func-preserves-cod*:
 preserves-cod (*id-func* *AA*)
 ⟨*proof*⟩

lemma (in *one-cat*) *id-func-preserves-id*:
 preserves-id (*id-func* *AA*)
 ⟨*proof*⟩

lemma (in *one-cat*) *id-func-preserves-comp*:
 preserves-comp (*id-func* *AA*)
 ⟨*proof*⟩

theorem (in *one-cat*) *id-func-functor*:
 Functor (*id-func* *AA*) : *AA* \longrightarrow *AA*
 ⟨*proof*⟩

end

4 HomFunctors

theory *HomFunctors*
imports *SetCat Functors*
begin

locale *into-set* = *two-cats* +
 assumes *AA* = (*AA*::('o,'a,'m)*category-scheme*)
 fixes *U* and *Set*
 defines *U* \equiv (*UNIV*::'a *set*)
 defines *Set* \equiv *set-cat* *U*
 assumes *BB-Set*: *BB* = *Set*
 fixes *homf* (*Hom*'(-,'-'))

defines $homf\ A \equiv \langle \langle$
 $om = (\lambda B \in Ob. Hom\ A\ B),$
 $am = (\lambda f \in Ar. \langle \langle set-dom = Hom\ A\ (Dom\ f), set-func = (\lambda g \in Hom\ A\ (Dom\ f). f \cdot$
 $g \rangle, set-cod = Hom\ A\ (Cod\ f) \rangle \rangle)$
 \rangle

lemma (**in** *into-set*) *homf-preserves-arrows*:
 $Hom(A, -)_a : Ar \rightarrow ar\ Set$
 $\langle proof \rangle$

lemma (**in** *into-set*) *homf-preserves-objects*:
 $Hom(A, -)_o : Ob \rightarrow ob\ Set$
 $\langle proof \rangle$

lemma (**in** *into-set*) *homf-preserves-dom*:
assumes $f: f \in Ar$
shows $Hom(A, -)_o (Dom\ f) = dom\ Set\ (Hom(A, -)_a\ f)$
 $\langle proof \rangle$

lemma (**in** *into-set*) *homf-preserves-cod*:
assumes $f: f \in Ar$
shows $Hom(A, -)_o (Cod\ f) = cod\ Set\ (Hom(A, -)_a\ f)$
 $\langle proof \rangle$

lemma (**in** *into-set*) *homf-preserves-id*:
assumes $B: B \in Ob$
shows $Hom(A, -)_a (Id\ B) = id\ Set\ (Hom(A, -)_o\ B)$
 $\langle proof \rangle$

lemma (**in** *into-set*) *homf-preserves-comp*:
assumes $f: f \in Ar$
and $g: g \in Ar$
and $fg: Cod\ f = Dom\ g$
shows $Hom(A, -)_a (g \cdot f) = (Hom(A, -)_a\ g) \odot (Hom(A, -)_a\ f)$
 $\langle proof \rangle$

theorem (**in** *into-set*) *homf-into-set*:
 $Functor\ Hom(A, -) : AA \longrightarrow Set$
 $\langle proof \rangle$

end

5 Natural Transformations

```
theory NatTrans
imports Functors
begin
```

```
locale natural-transformation = two-cats +
  fixes F and G and u
  assumes Functor F : AA → BB
  and Functor G : AA → BB
  and u : ob AA → ar BB
  and u ∈ extensional (ob AA)
  and ∀ A ∈ Ob. u A ∈ Hom2 (Fo A) (Go A)
  and ∀ A ∈ Ob. ∀ B ∈ Ob. ∀ f ∈ Hom A B. (Ga f) •2 (u A) = (u B) •2 (Fa f)
```

abbreviation

```
nt-syn (- : - ⇒ - in Func '( - , - ') [81]) where
u : F ⇒ G in Func(AA, BB) ≡ natural-transformation AA BB F G u
```

```
locale endoNT = natural-transformation + one-cat
```

theorem (in endoNT) *id-restrict-natural*:

```
(λA ∈ Ob. Id A) : (id-func AA) ⇒ (id-func AA) in Func(AA, AA)
⟨proof⟩
```

end

6 Yoneda Lemma

```
theory Yoneda
imports HomFunctors NatTrans
begin
```

6.1 The Sandwich Natural Transformation

```
locale Yoneda = functor + into-set +
  assumes AA = (AA :: ('o, 'a, 'm) category-scheme)
  fixes sandwich :: ['o, 'a, 'o] ⇒ 'a set-arrow (σ'(-, -'))
  defines sandwich A a ≡ (λB ∈ Ob. (
    set-dom = Hom A B,
    set-func = (λf ∈ Hom A B. set-func (Fa f) a),
    set-cod = Fo B
  ))
  fixes unsandwich :: ['o, 'o] ⇒ 'a set-arrow ⇒ 'a (σ←'(-, -'))
  defines unsandwich A u ≡ set-func (u A) (Id A)
```

lemma (in Yoneda) *F*-into-set:

Functor $F : AA \longrightarrow Set$

<proof>

lemma (in Yoneda) *F*-comp-func:

assumes 1: $A \in Ob$ **and** 2: $B \in Ob$ **and** 3: $C \in Ob$

and 4: $g \in Hom\ A\ B$ **and** 5: $f \in Hom\ B\ C$

shows $set\text{-}func\ (F_a\ (f \cdot g)) = compose\ (F_o\ A)\ (set\text{-}func\ (F_a\ f))\ (set\text{-}func\ (F_a\ g))$

<proof>

lemma (in Yoneda) sandwich-funcset:

assumes $A: A \in Ob$

and $a \in F_o\ A$

shows $\sigma(A,a) : Ob \rightarrow ar\ Set$

<proof>

lemma (in Yoneda) sandwich-type:

assumes $A: A \in Ob$ **and** $B: B \in Ob$

and $a \in F_o\ A$

shows $\sigma(A,a)\ B \in hom\ Set\ (Hom\ A\ B)\ (F_o\ B)$

<proof>

lemma (in Yoneda) sandwich-commutes:

assumes $AOb: A \in Ob$ **and** $BOb: B \in Ob$ **and** $COb: C \in Ob$

and $aFa: a \in F_o\ A$

and $fBC: f \in Hom\ B\ C$

shows $(F_a\ f) \odot (\sigma(A,a)\ B) = (\sigma(A,a)\ C) \odot (Hom(A,-)_a\ f)$

<proof>

lemma (in Yoneda) sandwich-natural:

assumes $A \in Ob$

and $a \in F_o\ A$

shows $\sigma(A,a) : Hom(A,-) \Rightarrow F\ in\ Func(AA,Set)$

<proof>

6.2 Sandwich Components are Bijective

lemma (in Yoneda) unsandwich-left-inverse:

assumes 1: $A \in Ob$

and 2: $a \in F_o\ A$

shows $\sigma^{\leftarrow}(A,\sigma(A,a)) = a$

<proof>

lemma (in *Yoneda*) *unsandwich-right-inverse*:

assumes 1: $A \in Ob$

and 2: $u : Hom(A, -) \Rightarrow F$ in $Func(AA, Set)$

shows $\sigma(A, \sigma^{\leftarrow}(A, u)) = u$

<proof>

In order to state the lemma, we must rectify a curious omission from the Isabelle/HOL library. They define the idea of injectivity on a given set, but surjectivity is only defined relative to the entire universe of the target type.

definition

surj-on :: [$'a \Rightarrow 'b$, $'a\ set$, $'b\ set$] $\Rightarrow bool$ **where**

surj-on $f\ A\ B \iff (\forall y \in B. \exists x \in A. f(x)=y)$

definition

bij-on :: [$'a \Rightarrow 'b$, $'a\ set$, $'b\ set$] $\Rightarrow bool$ **where**

bij-on $f\ A\ B \iff inj-on\ f\ A \ \& \ surj-on\ f\ A\ B$

definition

equinumerous :: [$'a\ set$, $'b\ set$] $\Rightarrow bool$ (**infix** \cong 40) **where**

equinumerous $A\ B \iff (\exists f. bij-on\ f\ A\ B)$

theorem (in *Yoneda*) *Yoneda*:

assumes 1: $A \in Ob$

shows $F \circ A \cong \{u. u : Hom(A, -) \Rightarrow F \text{ in } Func(AA, Set)\}$

<proof>

end

References

- [O’K04] Greg O’Keefe. Towards a readable formalisation of category theory. In Mike Atkinson, editor, *Computing: The Australasian Theory Symposium*, volume 91 of *Electronic Notes in Theoretical Computer Science*, pages 212–228. Elsevier, 2004.