

Formalization of a Generalized Protocol for Clock Synchronization in Isabelle/HOL

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April 29, 2009

Abstract

We formalize the generalized Byzantine fault-tolerant clock synchronization protocol of Schneider. This protocol abstracts from particular algorithms or implementations for clock synchronization. This abstraction includes several assumptions on the behaviors of physical clocks and on general properties of concrete algorithms/implementations. Based on these assumptions the correctness of the protocol is proved by Schneider. His proof was later verified by Shankar using the theorem prover EHDm (precursor to PVS). Our formalization in Isabelle/HOL is based on Shankar’s formalization.

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1 Introduction

In certain distributed systems, e.g., real-time process-control systems, the existence of a reliable global time source is critical in ensuring the correct functioning of the systems. This reliable global time source can be implemented using several physical clocks distributed on different nodes in the distributed system. Since physical clocks are by nature constantly drifting away from the “real time” and different clocks can have different drift rates, in such a scheme, it is important that these clocks are regularly adjusted so that they are closely synchronized within a certain application-specific safe bound. The design and verification of clock synchronization protocols are often complicated by the additional requirement that the protocols should work correctly under certain types of errors, e.g., failure of some clocks, error in communication network or corrupted messages, etc.

There has been a number of fault-tolerant clock synchronization algorithms studied in the literature, e.g., the *Interactive Convergence Algorithm* (ICA) by Lamport and Melliar-Smith [1], the Lundelius-Lynch algorithm [2], etc., each with its own degree of fault tolerance. One important property that must be satisfied by a clock synchronization algorithm is the agreement property, i.e., at any time t , the difference of the clock readings of any two non-faulty processes must be bounded by a constant (which is fixed according to the domain of applications). At the core of these algorithms is the convergence function that calculates the adjustment to a clock of a process, based on the clock readings of all other processes. Schneider [3] gives an abstract characterization of a wide range of clock synchronization algorithms (based on the convergence functions used) and proves the agreement property in this abstract framework. Schneider's proof was later verified by Shankar [4] in the theorem prover EHDM (precursor to PVS), where eleven axioms about clocks are explicitly stated.

We formalize Schneider's proof in Isabelle/HOL, making use of Shankar's formulation of the clock axioms. The particular formulation of axioms on clock conditions and the statements of the main theorems here are essentially those of Shankar's [4], with some minor changes in syntax. For the full description of the protocol, the general structure of the proof and the meaning of the constants and function symbols used in this formalization, we refer readers to [4].

Acknowledgment I would like to thank Stephan Merz and Pascal Fontaine for useful tips on using Isabelle and particularly the Isar proof language.

2 Isar proof scripts

```
theory GenClock imports Complex-Main begin
```

2.1 Preliminary lemmas

```
lemma AddMin:  $\forall (x::real) (y::real). x + -y = x - y$  by (simp)
```

```
lemma AbsSym:  $abs ( (x::real) - (y::real) ) = abs (y - x)$   
by (simp add: real-abs-def)
```

```
lemma Factor1:  $(x :: real) * (y::real) - y = y * (x - 1)$ 
```

```
proof -
```

```
  have  $(x + -1) * y = y * (x - 1)$   
  by (simp add: real-mult-commute AddMin)  
  thus  $x * y - y = y * (x - 1)$   
  by (simp add: real-add-mult-distrib)
```

```
qed
```

```
lemma Factor2:  $(x::real) * (1 - (y::real)) = x - x * y$ 
```

```
proof -
```

```
  from real-add-mult-distrib have  $x * (1 + -y) = x + -x * y$   
  by (auto simp add: real-mult-commute)  
  thus ?thesis by (simp add: AddMin)
```

```
qed
```

```

lemma obvious: (x :: real)*(1 + y) - x*(1 - y) = 2*x*y
proof-
  have x*(1 + y) = x + x*y
  proof-
    have x*(1 + y) = (1 + y)*x
      by(simp add: real-mult-commute)
    moreover
    have (1 + y)*x = x + y*x
      by (simp add: real-add-mult-distrib)
    moreover
    have x + y*x = x + x*y by simp
    ultimately show ?thesis by simp
  qed
  moreover
  have x*(1 - y) = x - x*y
  proof-
    have x*(1 - y) = (1 - y)*x by simp
    moreover
    have (1 - y)*x = (1 + -y)*x by simp
    moreover
    have (1 + -y)*x = x - y*x by (simp add: real-add-mult-distrib)
    ultimately show ?thesis by simp
  qed
  ultimately show ?thesis by simp
qed

```

2.2 Types and constants definitions

Process is represented by natural numbers. The type 'event' corresponds to synchronization rounds.

types

```

process = nat
event = nat
time = real
Clocktime = real

```

axiomatization

```

δ :: real and
μ :: real and
ρ :: real and
rmin :: real and
rmax :: real and
β :: real and
Λ :: real and

```

```

np :: process and
maxfaults :: process and

```

```

PC :: [process, time] ⇒ Clocktime and

```

```

VC :: [process, time] ⇒ Clocktime and

```

```

te :: [process, event] ⇒ time and

```

$\vartheta :: [\text{process}, \text{event}] \Rightarrow (\text{process} \Rightarrow \text{Clocktime})$ **and**

$IC :: [\text{process}, \text{event}, \text{time}] \Rightarrow \text{Clocktime}$ **and**

$\text{correct} :: [\text{process}, \text{time}] \Rightarrow \text{bool}$ **and**

$\text{cfn} :: [\text{process}, (\text{process} \Rightarrow \text{Clocktime})] \Rightarrow \text{Clocktime}$ **and**

$\pi :: [\text{Clocktime}, \text{Clocktime}] \Rightarrow \text{Clocktime}$ **and**

$\alpha :: \text{Clocktime} \Rightarrow \text{Clocktime}$

definition

$\text{count} :: [\text{process} \Rightarrow \text{bool}, \text{process}] \Rightarrow \text{nat}$ **where**
 $\text{count } f \ n = \text{card } \{p. p < n \wedge f \ p\}$

definition

$\text{Adj} :: [\text{process}, \text{event}] \Rightarrow \text{Clocktime}$ **where**
 $\text{Adj} = (\lambda p \ i. \text{if } 0 < i \text{ then } \text{cfn } p \ (\vartheta \ p \ i) - \text{PC } p \ (\text{te } p \ i)$
 $\text{else } 0)$

definition

$\text{okRead1} :: [\text{process} \Rightarrow \text{Clocktime}, \text{Clocktime}, \text{process} \Rightarrow \text{bool}] \Rightarrow \text{bool}$ **where**
 $\text{okRead1 } f \ x \ \text{ppred} \longleftrightarrow (\forall l \ m. \text{ppred } l \wedge \text{ppred } m \longrightarrow |f \ l - f \ m| \leq x)$

definition

$\text{okRead2} :: [\text{process} \Rightarrow \text{Clocktime}, \text{process} \Rightarrow \text{Clocktime}, \text{Clocktime},$
 $\text{process} \Rightarrow \text{bool}] \Rightarrow \text{bool}$ **where**
 $\text{okRead2 } f \ g \ x \ \text{ppred} \longleftrightarrow (\forall p. \text{ppred } p \longrightarrow |f \ p - g \ p| \leq x)$

definition

$\text{rho-bound1} :: [[\text{process}, \text{time}] \Rightarrow \text{Clocktime}] \Rightarrow \text{bool}$ **where**
 $\text{rho-bound1 } C \longleftrightarrow (\forall p \ s \ t. \text{correct } p \ t \wedge s \leq t \longrightarrow C \ p \ t - C \ p \ s \leq (t - s) * (1 + \varrho))$

definition

$\text{rho-bound2} :: [[\text{process}, \text{time}] \Rightarrow \text{Clocktime}] \Rightarrow \text{bool}$ **where**
 $\text{rho-bound2 } C \longleftrightarrow (\forall p \ s \ t. \text{correct } p \ t \wedge s \leq t \longrightarrow (t - s) * (1 - \varrho) \leq C \ p \ t - C \ p \ s)$

2.3 Clock conditions

Some general assumptions

axioms

$\text{constants-ax: } 0 < \beta \wedge 0 < \mu \wedge 0 < rmin$
 $\wedge rmin \leq rmax \wedge 0 < \varrho \wedge 0 < np \wedge \text{maxfaults} \leq np$

$\text{PC-monotone: } \forall p \ s \ t. \text{correct } p \ t \wedge s \leq t \longrightarrow \text{PC } p \ s \leq \text{PC } p \ t$

$\text{VClock: } \forall p \ t \ i. \text{correct } p \ t \wedge \text{te } p \ i \leq t \wedge t < \text{te } p \ (i + 1) \longrightarrow \text{VC } p \ t = \text{IC } p \ i \ t$

$\text{IClock: } \forall p \ t \ i. \text{correct } p \ t \longrightarrow \text{IC } p \ i \ t = \text{PC } p \ t + \text{Adj } p \ i$

Condition 1: initial skew

axioms

$$\text{init}: \forall p. \text{correct } p \ 0 \longrightarrow 0 \leq PC \ p \ 0 \wedge PC \ p \ 0 \leq \mu$$

Condition 2: bounded drift

axioms

$$\begin{aligned} \text{rate-1}: & \forall p \ s \ t. \text{correct } p \ t \wedge s \leq t \longrightarrow PC \ p \ t - PC \ p \ s \leq (t - s) * (1 + \varrho) \\ \text{rate-2}: & \forall p \ s \ t. \text{correct } p \ t \wedge s \leq t \longrightarrow (t - s) * (1 - \varrho) \leq PC \ p \ t - PC \ p \ s \end{aligned}$$

Condition 3: bounded interval

axioms

$$\begin{aligned} \text{rts0}: & \forall p \ t \ i. \text{correct } p \ t \wedge t \leq te \ p \ (i+1) \longrightarrow t - te \ p \ i \leq rmax \\ \text{rts1}: & \forall p \ t \ i. \text{correct } p \ t \wedge te \ p \ (i+1) \leq t \longrightarrow rmin \leq t - te \ p \ i \end{aligned}$$

Condition 4 : bounded delay

axioms

$$\begin{aligned} \text{rts2a}: & \forall p \ q \ t \ i. \text{correct } p \ t \wedge \text{correct } q \ t \wedge te \ q \ i + \beta \leq t \longrightarrow te \ p \ i \leq t \\ \text{rts2b}: & \forall p \ q \ i. \text{correct } p \ (te \ p \ i) \wedge \text{correct } q \ (te \ q \ i) \longrightarrow abs(te \ p \ i - te \ q \ i) \leq \beta \end{aligned}$$

Condition 5: initial synchronization

axioms

$$\text{synch0}: \forall p. te \ p \ 0 = 0$$

Condition 6: nonoverlap

axioms

$$\text{nonoverlap}: \beta \leq rmin$$

Condition 7: reading errors

axioms

$$\begin{aligned} \text{readerror}: & \forall p \ q \ i. \text{correct } p \ (te \ p \ (i+1)) \wedge \text{correct } q \ (te \ p \ (i+1)) \longrightarrow \\ & abs(\vartheta \ p \ (i+1) \ q - IC \ q \ i \ (te \ p \ (i+1))) \leq \Lambda \end{aligned}$$

Condition 8: bounded faults

axioms

$$\begin{aligned} \text{correct-closed}: & \forall p \ s \ t. s \leq t \wedge \text{correct } p \ t \longrightarrow \text{correct } p \ s \\ \text{correct-count}: & \forall t. np - maxfaults \leq count \ (\lambda p. \text{correct } p \ t) \ np \end{aligned}$$

Condition 9: Translation invariance

axioms

$$\text{trans-inv}: \forall p \ f \ x. 0 \leq x \longrightarrow cf_n \ p \ (\lambda y. f \ y + x) = cf_n \ p \ f + x$$

Condition 10: precision enhancement

axioms

$$\begin{aligned} \text{prec-enh}: & \\ \forall \text{ppred } p \ q \ f \ g \ x \ y. & \\ np - maxfaults \leq count \ \text{ppred } np \wedge & \\ okRead1 \ f \ y \ \text{ppred} \wedge okRead1 \ g \ y \ \text{ppred} \wedge & \\ okRead2 \ f \ g \ x \ \text{ppred} \wedge \text{ppred } p \wedge \text{ppred } q & \\ \longrightarrow abs(cf_n \ p \ f - cf_n \ q \ g) \leq \pi \ x \ y & \end{aligned}$$

Condition 11: accuracy preservation

axioms

acc-prsv:
 $\forall \text{ ppred } p \ q \ f \ x. \text{ okRead1 } f \ x \ \text{ppred} \wedge \text{ np} - \text{maxfaults} \leq \text{count } \text{ppred } \text{np}$
 $\wedge \text{ ppred } p \wedge \text{ ppred } q \longrightarrow \text{abs}(\text{cfn } p \ f - f \ q) \leq \alpha \ x$

2.3.1 Some derived properties of clocks

lemma *rts0d:*
assumes *cp: correct p (te p (i+1))*
shows $\text{te } p \ (i+1) - \text{te } p \ i \leq \text{rmax}$
using *cp rts0* **by** *simp*

lemma *rts1d:*
assumes *cp: correct p (te p (i+1))*
shows $\text{rmin} \leq \text{te } p \ (i+1) - \text{te } p \ i$
using *cp rts1* **by** *simp*

lemma *rte:*
assumes *cp: correct p (te p (i+1))*
shows $\text{te } p \ i \leq \text{te } p \ (i+1)$
proof–
from *cp rts1d* **have** $\text{rmin} \leq \text{te } p \ (i+1) - \text{te } p \ i$
by *simp*
from *this constants-ax* **show** *?thesis* **by** *arith*
qed

lemma *beta-bound1:*
assumes *corr-p: correct p (te p (i+1))*
and *corr-q: correct q (te p (i+1))*
shows $0 \leq \text{te } p \ (i+1) - \text{te } q \ i$
proof–
from *corr-p rte* **have** $\text{te } p \ i \leq \text{te } p \ (i+1)$
by *simp*
from *this corr-p correct-closed* **have** *corr-pi: correct p (te p i)*
by *blast*
from *corr-p rts1d nonoverlap* **have** $\text{rmin} \leq \text{te } p \ (i+1) - \text{te } p \ i$
by *simp*
from *this nonoverlap* **have** $\beta \leq \text{te } p \ (i+1) - \text{te } p \ i$ **by** *simp*
hence $\text{te } p \ i + \beta \leq \text{te } p \ (i+1)$ **by** *simp*

from *this corr-p corr-q rts2a*
have $\text{te } q \ i \leq \text{te } p \ (i+1)$
by *blast*
thus *?thesis* **by** *simp*
qed

lemma *beta-bound2:*
assumes *corr-p: correct p (te p (i+1))*
and *corr-q: correct q (te q i)*
shows $\text{te } p \ (i+1) - \text{te } q \ i \leq \text{rmax} + \beta$
proof–
from *corr-p rte* **have** $\text{te } p \ i \leq \text{te } p \ (i+1)$
by *simp*
from *this corr-p correct-closed* **have** *corr-pi: correct p (te p i)*

by *blast*

have *split*: $te\ p\ (i+1) - te\ q\ i =$
 $(te\ p\ (i+1) - te\ p\ i) + (te\ p\ i - te\ q\ i)$
by (*simp*)

from *corr-q corr-pi rts2b* have *Eq1*: $abs(te\ p\ i - te\ q\ i) \leq \beta$

by *simp*

have *Eq2*: $te\ p\ i - te\ q\ i \leq \beta$

proof *cases*

assume $te\ q\ i \leq te\ p\ i$

from *this Eq1* show *?thesis*

by (*simp add: real-abs-def*)

next

assume $\neg (te\ q\ i \leq te\ p\ i)$

from *this Eq1* show *?thesis*

by (*simp add: real-abs-def*)

qed

from *corr-p rts0d* have $te\ p\ (i+1) - te\ p\ i \leq rmax$

by *simp*

from *this split Eq2* show *?thesis* by *simp*

qed

2.3.2 Bounded-drift for logical clocks (IC)

lemma *bd*:

assumes *ie*: $s \leq t$

and *rb1*: *rho-bound1* *C*

and *rb2*: *rho-bound2* *D*

and *PC-ie*: $D\ q\ t - D\ q\ s \leq C\ p\ t - C\ p\ s$

and *corr-p*: *correct* *p* *t*

and *corr-q*: *correct* *q* *t*

shows $|C\ p\ t - D\ q\ t| \leq |C\ p\ s - D\ q\ s| + 2*\rho*(t - s)$

proof–

let *?Dt* = $C\ p\ t - D\ q\ t$

let *?Ds* = $C\ p\ s - D\ q\ s$

let *?Bp* = $C\ p\ t - C\ p\ s$

let *?Bq* = $D\ q\ t - D\ q\ s$

let *?I* = $t - s$

have $|?Bp - ?Bq| \leq 2*\rho*(t - s)$

proof–

from *PC-ie* have *Eq1*: $|?Bp - ?Bq| = ?Bp - ?Bq$ by (*simp add: real-abs-def*)

from *corr-p ie rb1* have *Eq2*: $?Bp - ?Bq \leq ?I*(1+\rho) - ?Bq$ (**is** *?E1* \leq *?E2*)

by (*simp add: AddMin rho-bound1-def*)

from *corr-q ie rb2* have $?I*(1 - \rho) \leq ?Bq$

by (*simp add: rho-bound2-def*)

from *this* have *Eq3*: $?E2 \leq ?I*(1+\rho) - ?I*(1 - \rho)$

by (*simp*)

have *Eq4*: $?I*(1+\rho) - ?I*(1 - \rho) = 2*\rho*?I$

by (*simp add: obvious*)

from *Eq1 Eq2 Eq3 Eq4* show *?thesis* by *simp*

qed
 moreover
 have $|?Dt| \leq |?Bp - ?Bq| + |?Ds|$
 by(*simp add: real-abs-def*)
 ultimately show *?thesis* by *simp*
 qed

lemma *bounded-drift*:

assumes *ie*: $s \leq t$
 and *rb1*: *rho-bound1* *C*
 and *rb2*: *rho-bound2* *C*
 and *rb3*: *rho-bound1* *D*
 and *rb4*: *rho-bound2* *D*
 and *corr-p*: *correct* *p* *t*
 and *corr-q*: *correct* *q* *t*
 shows $|C\ p\ t - D\ q\ t| \leq |C\ p\ s - D\ q\ s| + 2*\varrho*(t - s)$

proof–

let $?Bp = C\ p\ t - C\ p\ s$
 let $?Bq = D\ q\ t - D\ q\ s$

show *?thesis*

proof *cases*

assume $?Bq \leq ?Bp$
 from *this ie rb1 rb4 corr-p corr-q bd* show *?thesis* by *simp*

next

assume $\neg (?Bq \leq ?Bp)$
 hence $?Bp \leq ?Bq$ by *simp*
 from *this ie rb2 rb3 corr-p corr-q bd*
 have $|D\ q\ t - C\ p\ t| \leq |D\ q\ s - C\ p\ s| + 2*\varrho*(t - s)$
 by *simp*
 from *this* show *?thesis* by (*simp add: AbsSym*)

qed

qed

Drift rate of logical clocks

lemma *IC-rate1*:

rho-bound1 $(\lambda\ p\ t.\ IC\ p\ i\ t)$

proof–

{
 fix *p*::*process*
 fix *s*::*time*
 fix *t*::*time*
 assume *cp*: *correct* *p* *t*
 assume *ie*: $s \leq t$
 from *cp ie correct-closed* have *cps*: *correct* *p* *s*
 by *blast*
 have $IC\ p\ i\ t - IC\ p\ i\ s \leq (t - s)*(1 + \varrho)$

proof–

from *cp IClock* have $IC\ p\ i\ t = PC\ p\ t + Adj\ p\ i$
 by *simp*

moreover

from *cps IClock* have $IC\ p\ i\ s = PC\ p\ s + Adj\ p\ i$
 by *simp*

```

moreover
from cp ie rate-1 have  $PC\ p\ t - PC\ p\ s \leq (t - s) * (1 + \rho)$ 
  by simp
ultimately show ?thesis by simp
qed
}
thus ?thesis by (simp add: rho-bound1-def)
qed

```

lemma *IC-rate2*:

rho-bound2 ($\lambda\ p\ t.\ IC\ p\ i\ t$)

proof–

```

{
  fix p::process
  fix s::time
  fix t::time
  assume cp: correct p t
  assume ie: s ≤ t
  from cp ie correct-closed have cps: correct p s
    by blast
  have  $(t - s) * (1 - \rho) \leq IC\ p\ i\ t - IC\ p\ i\ s$ 
  proof–
    from cp IClock have  $IC\ p\ i\ t = PC\ p\ t + Adj\ p\ i$ 
      by simp
    moreover
    from cps IClock have  $IC\ p\ i\ s = PC\ p\ s + Adj\ p\ i$ 
      by simp
    moreover
    from cp ie rate-2 have  $(t - s) * (1 - \rho) \leq PC\ p\ t - PC\ p\ s$ 
      by simp
    ultimately show ?thesis by simp
  qed
}
thus ?thesis by (simp add: rho-bound2-def)
qed

```

Auxiliary function *ICf*: we introduce this to avoid some unification problem in some tactic of *isabelle*.

definition

ICf :: *nat* \Rightarrow (*process* \Rightarrow *time* \Rightarrow *Clocktime*) **where**

ICf *i* = ($\lambda\ p\ t.\ IC\ p\ i\ t$)

lemma *IC-bd*:

assumes *ie: s ≤ t*

and *corr-p: correct p t*

and *corr-q: correct q t*

shows $|IC\ p\ i\ t - IC\ q\ j\ t| \leq |IC\ p\ i\ s - IC\ q\ j\ s| + 2 * \rho * (t - s)$

proof–

let *?C* = *ICf* *i*

let *?D* = *ICf* *j*

let *?G* = $|?C\ p\ t - ?D\ q\ t| \leq |?C\ p\ s - ?D\ q\ s| + 2 * \rho * (t - s)$

from *IC-rate1* **have** *rb1: rho-bound1 (ICf i) ∧ rho-bound1 (ICf j)*

by (simp add: ICf-def)
 from IC-rate2 have rb2: rho-bound2 (ICf i) \wedge rho-bound2 (ICf j)
 by (simp add: ICf-def)
 from ie rb1 rb2 corr-p corr-q bounded-drift
 have ?G by simp
 from this show ?thesis by (simp add: ICf-def)
 qed

lemma event-bound:
assumes ie1: $0 \leq (t::real)$
and corr-p: correct p t
and corr-q: correct q t
shows $\exists i. t < \max (te\ p\ i) (te\ q\ i)$
proof (rule ccontr)
 assume A: $\neg (\exists i. t < \max (te\ p\ i) (te\ q\ i))$
 show False
proof–
 have F1: $\forall i. te\ p\ i \leq t$
proof
 fix i :: nat
 from A have $\neg (t < \max (te\ p\ i) (te\ q\ i))$
 by simp
 hence Eq1: $\max (te\ p\ i) (te\ q\ i) \leq t$ by arith
 have Eq2: $te\ p\ i \leq \max (te\ p\ i) (te\ q\ i)$
 by (simp add: max-def)
 from Eq1 Eq2 show $te\ p\ i \leq t$ by simp
 qed

have F2: $\forall (i :: nat). \text{correct } p (te\ p\ i)$
proof
 fix i :: nat
 from F1 have $te\ p\ i \leq t$ by simp
 from this corr-p correct-closed
 show correct p (te p i) by blast
 qed

have F3: $\forall (i :: nat). \text{real } i * rmin \leq te\ p\ i$
proof
 fix i :: nat
 show $\text{real } i * rmin \leq te\ p\ i$
proof (induct i)
 from synch0 show $\text{real } (0::nat) * rmin \leq te\ p\ 0$ by simp
 next
 fix i :: nat assume ind-hyp: $\text{real } i * rmin \leq te\ p\ i$

 show $\text{real } (Suc\ i) * rmin \leq te\ p\ (Suc\ i)$
proof–

 have Eq1: $\text{real } i * rmin + rmin = (\text{real } i + 1) * rmin$
 by (simp add: real-add-mult-distrib)

have $Eq2: \text{real } i + 1 = \text{real } (i+1)$ **by** *simp*
from $Eq1$ $Eq2$
have $Eq3: \text{real } i * rmin + rmin = \text{real } (i+1) * rmin$
by(*simp*)

from $F2$ **have** $cp1: \text{correct } p (te\ p\ (i+1))$
by *simp*
from $F2$ **have** $cp2: \text{correct } p (te\ p\ i)$
by *simp*
from $cp1$ $rts1d$ **have** $rmin \leq te\ p\ (i+1) - te\ p\ i$
by *simp*
hence $Eq4: te\ p\ i + rmin \leq te\ p\ (i+1)$ **by** *simp*
from *ind-hyp* **have** $\text{real } i * rmin + rmin \leq te\ p\ i + rmin$
by (*simp*)
from *this* $Eq4$ **have** $\text{real } i * rmin + rmin \leq te\ p\ (i+1)$
by *simp*
from *this* $Eq3$ **show** *?thesis* **by** *simp*
qed
qed
qed

have $F4: \forall (i::nat). \text{real } i * rmin \leq t$
proof
fix $i::nat$
from $F1$ **have** $te\ p\ i \leq t$ **by** *simp*
moreover
from $F3$ **have** $\text{real } i * rmin \leq te\ p\ i$ **by** *simp*
ultimately show $\text{real } i * rmin \leq t$ **by** *simp*
qed

from *constants-ax* **have** $0 < rmin$ **by** *simp*

from *this* *reals-Archimedean3*
have $Archi: \exists (k::nat). t < \text{real } k * rmin$
by *blast*

from $Archi$ **obtain** $k::nat$ **where** $C: t < \text{real } k * rmin ..$

from $F4$ **have** $\text{real } k * rmin \leq t$ **by** *simp*
hence $notC: \neg (t < \text{real } k * rmin)$ **by** *simp*

from C $notC$ **show** *False* **by** *simp*
qed
qed

2.4 Agreement property

definition $\gamma1\ x = \pi (2 * \rho * \beta + 2 * \Lambda) (2 * \Lambda + x + 2 * \rho * (rmax + \beta))$

definition $\gamma2\ x = x + 2 * \rho * rmax$

definition $\gamma3\ x = \alpha (2 * \Lambda + x + 2 * \rho * (rmax + \beta)) + \Lambda + 2 * \rho * \beta$

definition

$okmaxsync :: [nat, Clocktime] \Rightarrow bool$ **where**

$$\begin{aligned}
okmaxsync\ i\ x &\longleftrightarrow (\forall\ p\ q.\ correct\ p\ (\max\ (te\ p\ i)\ (te\ q\ i)) \\
&\wedge\ correct\ q\ (\max\ (te\ p\ i)\ (te\ q\ i)) \longrightarrow \\
&|IC\ p\ i\ (\max\ (te\ p\ i)\ (te\ q\ i)) - IC\ q\ i\ (\max\ (te\ p\ i)\ (te\ q\ i))| \leq x)
\end{aligned}$$

definition

$okClocks :: [process, process, nat] \Rightarrow bool$ **where**
 $okClocks\ p\ q\ i \longleftrightarrow (\forall\ t.\ 0 \leq t \wedge t < \max\ (te\ p\ i)\ (te\ q\ i)) \longrightarrow$
 $\wedge\ correct\ p\ t \wedge correct\ q\ t$
 $\longrightarrow |VC\ p\ t - VC\ q\ t| \leq \delta)$

lemma *okClocks-sym:*

assumes *ok-pq:* $okClocks\ p\ q\ i$

shows $okClocks\ q\ p\ i$

proof-

```

{
  fix t :: time
  assume ie1: 0 ≤ t
  assume ie2: t < max (te q i) (te p i)
  assume corr-q: correct q t
  assume corr-p: correct p t

  have max (te q i) (te p i) = max (te p i) (te q i)
    by (simp add: max-def)
  from this ok-pq ie1 ie2 corr-p corr-q
  have |VC q t - VC p t| ≤ δ
    by (simp add: AbsSym okClocks-def)
}
thus ?thesis by (simp add: okClocks-def)
qed

```

lemma *ICp-Suc:*

assumes *corr-p:* $correct\ p\ (te\ p\ (i+1))$

shows $IC\ p\ (i+1)\ (te\ p\ (i+1)) = cfn\ p\ (\vartheta\ p\ (i+1))$

using *corr-p* *IClock* **by**(*simp add: Adj-def*)

lemma *IC-trans-inv:*

assumes *ie1:* $te\ q\ (i+1) \leq te\ p\ (i+1)$

and *corr-p:* $correct\ p\ (te\ p\ (i+1))$

and *corr-q:* $correct\ q\ (te\ p\ (i+1))$

shows

$IC\ q\ (i+1)\ (te\ p\ (i+1)) =$
 $cfn\ q\ (\lambda\ n.\ \vartheta\ q\ (i+1)\ n + (PC\ q\ (te\ p\ (i+1)) - PC\ q\ (te\ q\ (i+1))))$
(is *?T1 = ?T2***)**

proof-

let *?X* = $PC\ q\ (te\ p\ (i+1)) - PC\ q\ (te\ q\ (i+1))$

from *corr-q* *ie1* *PC-monotone* **have** *posX:* $0 \leq ?X$

by (*simp add: le-diff-eq*)

from *IClock* *corr-q* **have** *?T1 = cfn* $q\ (\vartheta\ q\ (i+1)) + ?X$

by(*simp add: Adj-def*)

from *this* *posX* *trans-inv* **show** *?thesis* **by** *simp*

qed

lemma *beta-rho*:

assumes *ie*: $te\ q\ (i+1) \leq te\ p\ (i+1)$

and *corr-p*: *correct p (te p (i+1))*

and *corr-q*: *correct q (te p (i+1))*

and *corr-l*: *correct l (te p (i+1))*

shows $|(PC\ l\ (te\ p\ (i+1)) - PC\ l\ (te\ q\ (i+1))) - (te\ p\ (i+1) - te\ q\ (i+1))| \leq \beta * \varrho$

proof–

let $?X = (PC\ l\ (te\ p\ (i+1)) - PC\ l\ (te\ q\ (i+1)))$

let $?D = te\ p\ (i+1) - te\ q\ (i+1)$

from *ie* **have** *posD*: $0 \leq ?D$ **by** *simp*

from *ie* *PC-monotone corr-l* **have** *posX*: $0 \leq ?X$

by (*simp add: le-diff-eq*)

from *ie* *corr-l rate-1* **have** *bound1*: $?X \leq ?D * (1 + \varrho)$ **by** *simp*

from *ie* *corr-l correct-closed* **have** *corr-l-tq*: *correct l (te q (i+1))*

by (*blast*)

from *ie* *corr-q correct-closed* **have** *corr-q-tq*: *correct q (te q (i+1))*

by *blast*

from *corr-q-tq corr-p rts2b* **have** $|?D| \leq \beta$

by (*simp*)

from *this constants-ax posD* **have** *D-beta*: $?D * \varrho \leq \beta * \varrho$

by (*simp add: real-abs-def*)

show *?thesis*

proof *cases*

assume *A*: $?D \leq ?X$

from *posX posD A* **have** *absEq*: $|?X - ?D| = ?X - ?D$

by (*simp add: real-abs-def*)

from *bound1* **have** *bound2*: $?X - ?D \leq ?D * \varrho$

by (*simp add: real-mult-commute real-add-mult-distrib*)

from *D-beta absEq bound2* **show** *?thesis* **by** *simp*

next

assume *notA*: $\neg (?D \leq ?X)$

from *this* **have** *absEq2*: $|?X - ?D| = ?D - ?X$

by (*simp add: real-abs-def*)

from *ie corr-l rate-2* **have** *bound3*: $?D * (1 - \varrho) \leq ?X$ **by** *simp*

from *this* **have** $?D - ?X \leq ?D * \varrho$ **by** (*simp add: Factor2*)

from *this absEq2 D-beta* **show** *?thesis* **by** *simp*

qed

qed

This lemma (and the next one *pe-cond2*) proves an assumption used in the precision enhancement.

lemma *pe-cond1*:

assumes *ie*: $te\ q\ (i+1) \leq te\ p\ (i+1)$

and *corr-p*: *correct p (te p (i+1))*

and *corr-q*: *correct q (te p (i+1))*

and *corr-l*: *correct l (te p (i+1))*

shows $|\vartheta\ q\ (i+1)\ l + (PC\ q\ (te\ p\ (i+1)) - PC\ q\ (te\ q\ (i+1))) - \vartheta\ p\ (i+1)\ l| \leq 2 * \varrho * \beta + 2 * \Lambda$

(**is** $?M \leq ?N$)

proof–

let $?Xl = (PC\ l\ (te\ p\ (i+1)) - PC\ l\ (te\ q\ (i+1)))$
let $?Xq = (PC\ q\ (te\ p\ (i+1)) - PC\ q\ (te\ q\ (i+1)))$
let $?D = te\ p\ (i+1) - te\ q\ (i+1)$
let $?T = \vartheta\ p\ (i+1)\ l - \vartheta\ q\ (i+1)\ l$
let $?RE1 = \vartheta\ p\ (i+1)\ l - IC\ l\ i\ (te\ p\ (i+1))$
let $?RE2 = \vartheta\ q\ (i+1)\ l - IC\ l\ i\ (te\ q\ (i+1))$
let $?ICT = IC\ l\ i\ (te\ p\ (i+1)) - IC\ l\ i\ (te\ q\ (i+1))$

have $?M = |(?Xq - ?D) - (?T - ?D)|$
by(*simp add: real-abs-def*)

hence *Split*: $?M \leq |?Xq - ?D| + |?T - ?D|$
by(*simp add: real-abs-def*)

from *ie corr-q correct-closed* **have** *corr-q-tq: correct q (te q (i+1))*
by(*blast*)
from *ie corr-l correct-closed* **have** *corr-l-tq: correct l (te q (i+1))*
by *blast*

from *corr-p corr-q corr-l ie beta-rho*
have $XlD: |?Xl - ?D| \leq \beta * \varrho$
by *simp*

from *corr-p corr-q ie beta-rho*
have $XqD: |?Xq - ?D| \leq \beta * \varrho$ **by** *simp*

have $TD: |?T - ?D| \leq 2 * \Lambda + \beta * \varrho$

proof–

have $Eq1: |?T - ?D| = |(?T - ?ICT) + (?ICT - ?D)|$ (**is** $?E1 = ?E2$)
by (*simp add: real-abs-def*)

have $Eq2: ?E2 \leq |?T - ?ICT| + |?ICT - ?D|$
by(*simp add: real-abs-def*)

have $Eq3: |?T - ?ICT| \leq |?RE1| + |?RE2|$
by(*simp add: real-abs-def*)

from *readerror corr-p corr-l*
have $Eq4: |?RE1| \leq \Lambda$ **by** *simp*

from *corr-l-tq corr-q-tq this readerror*
have $Eq5: |?RE2| \leq \Lambda$ **by** *simp*

from $Eq3\ Eq4\ Eq5$ **have** $Eq6: |?T - ?ICT| \leq 2 * \Lambda$
by *simp*

have $Eq7: ?ICT - ?D = ?Xl - ?D$

proof–

from *corr-p rte* **have** $te\ p\ i \leq te\ p\ (i+1)$
by(*simp*)

from *this corr-l correct-closed* **have** *corr-l-tpi: correct l (te p i)*

by *blast*
from *corr-q-tq rte* **have** $te\ q\ i \leq te\ q\ (i+1)$
by *simp*
from *this corr-l-tq correct-closed* **have** *corr-l-tqi: correct l (te q i)*
by *blast*

from *IClock corr-l*
have $F1: IC\ l\ i\ (te\ p\ (i+1)) = PC\ l\ (te\ p\ (i+1)) + Adj\ l\ i$
by (*simp*)
from *IClock corr-l-tq*
have $F2: IC\ l\ i\ (te\ q\ (i+1)) = PC\ l\ (te\ q\ (i+1)) + Adj\ l\ i$
by *simp*
from $F1\ F2$ **show** *?thesis* **by** (*simp*)

qed

from *this XlD* **have** $Eq8: |\?ICT - \?D| \leq \beta * \varrho$
by *arith*
from $Eq1\ Eq2\ Eq6\ Eq8$ **show** *?thesis*
by (*simp*)

qed

from *Split XqD TD* **have** $F1: \?M \leq 2 * \beta * \varrho + 2 * \Lambda$
by (*simp*)
have $F2: 2 * \varrho * \beta + 2 * \Lambda = 2 * \beta * \varrho + 2 * \Lambda$
by *simp*
from $F1$ **show** *?thesis* **by** (*simp only: F2*)

qed

lemma *pe-cond2:*

assumes *ie: te m i ≤ te l i*
and *corr-k: correct k (te k (i+1))*
and *corr-l-tk: correct l (te k (i+1))*
and *corr-m-tk: correct m (te k (i+1))*
and *ind-hyp: |IC l i (te l i) - IC m i (te l i)| ≤ δS*
shows $|\vartheta\ k\ (i+1)\ l - \vartheta\ k\ (i+1)\ m| \leq 2 * \Lambda + \delta S + 2 * \varrho * (rmax + \beta)$

proof–

let $\?X = \vartheta\ k\ (i+1)\ l - \vartheta\ k\ (i+1)\ m$
let $\?N = 2 * \Lambda + \delta S + 2 * \varrho * (rmax + \beta)$
let $\?D1 = \vartheta\ k\ (i+1)\ l - IC\ l\ i\ (te\ k\ (i+1))$
let $\?D2 = \vartheta\ k\ (i+1)\ m - IC\ m\ i\ (te\ k\ (i+1))$
let $\?ICS = IC\ l\ i\ (te\ k\ (i+1)) - IC\ m\ i\ (te\ k\ (i+1))$
let $\?tlm = te\ l\ i$
let $\?IC = IC\ l\ i\ \?tlm - IC\ m\ i\ \?tlm$

have $Eq1: |\?X| = |(\?D1 - \?D2) + \?ICS|$ (**is** $\?E1 = \?E2$)
by (*simp add: real-abs-def*)

have $Eq2: \?E2 \leq |\?D1 - \?D2| + |\?ICS|$ **by** (*simp add: real-abs-def*)

from *corr-l-tk corr-k beta-bound1* **have** *ie-lk: te l i ≤ te k (i+1)*
by (*simp add: le-diff-eq*)

from *this corr-l-tk correct-closed* **have** *corr-l: correct l (te l i)*
by *blast*

from *ie-lk corr-l-tk corr-m-tk IC-bd*
have *Eq3: $|?ICS| \leq |?IC| + 2*\varrho*(te\ k\ (i+1) - ?tIm)$*
by *simp*

from *this ind-hyp* **have** *Eq4: $|?ICS| \leq \delta S + 2*\varrho*(te\ k\ (i+1) - ?tIm)$*
by *simp*

from *corr-l corr-k beta-bound2* **have** *$te\ k\ (i+1) - ?tIm \leq rmax + \beta$*
by *simp*

from *this constants-ax* **have** *$2*\varrho*(te\ k\ (i+1) - ?tIm) \leq 2*\varrho*(rmax + \beta)$*
by (*simp add: real-mult-le-cancel-iff2*)

from *this Eq4* **have** *Eq4a: $|?ICS| \leq \delta S + 2*\varrho*(rmax + \beta)$*
by (*simp*)

from *corr-k corr-l-tk readerror*
have *Eq5: $|?D1| \leq \Lambda$* **by** *simp*

from *corr-k corr-m-tk readerror*
have *Eq6: $|?D2| \leq \Lambda$* **by** *simp*

have *$|?D1 - ?D2| \leq |?D1| + |?D2|$* **by** (*simp add: real-abs-def*)

from *this Eq5 Eq6* **have** *Eq7: $|?D1 - ?D2| \leq 2*\Lambda$*
by (*simp*)

from *Eq1 Eq2 Eq4a Eq7 split* **show** *?thesis* **by** (*simp*)

qed

lemma *theta-bound*:

assumes *corr-l: correct l (te p (i+1))*

and *corr-m: correct m (te p (i+1))*

and *corr-p: correct p (te p (i+1))*

and *IC-bound*:

$$\begin{aligned}
& |IC\ l\ i\ (max\ (te\ l\ i)\ (te\ m\ i)) - IC\ m\ i\ (max\ (te\ l\ i)\ (te\ m\ i))| \\
& \leq \delta S
\end{aligned}$$

shows *$|\vartheta\ p\ (i+1)\ l - \vartheta\ p\ (i+1)\ m|$*
 $\leq 2*\Lambda + \delta S + 2*\varrho*(rmax + \beta)$

proof–

from *corr-p corr-l beta-bound1* **have** *tli-le-tp: $te\ l\ i \leq te\ p\ (i+1)$*
by (*simp add: le-diff-eq*)

from *corr-p corr-m beta-bound1* **have** *tmi-le-tp: $te\ m\ i \leq te\ p\ (i+1)$*
by (*simp add: le-diff-eq*)

let *?tml = $max\ (te\ l\ i)\ (te\ m\ i)$*

from *tli-le-tp tmi-le-tp* **have** *tml-le-tp: $?tml \leq te\ p\ (i+1)$*
by *simp*

from *tml-le-tp corr-l correct-closed* **have** *corr-l-tml: correct l ?tml*
by *blast*

from *tml-le-tp corr-m correct-closed* **have** *corr-m-tml: correct m ?tml*
by *blast*

let *?Y = $2*\Lambda + \delta S + 2*\varrho*(rmax + \beta)$*

show *$|\vartheta\ p\ (i+1)\ l - \vartheta\ p\ (i+1)\ m| \leq ?Y$*

proof cases
assume A : $te\ m\ i < te\ l\ i$

from *this IC-bound*
have $|IC\ l\ i\ (te\ l\ i) - IC\ m\ i\ (te\ l\ i)| \leq \delta S$
by (*simp add: max-def*)
from *this A corr-p corr-l corr-m pe-cond2*
show *?thesis* **by** (*simp*)
next
assume $\neg (te\ m\ i < te\ l\ i)$
hence $Eq1$: $te\ l\ i \leq te\ m\ i$ **by** *simp*
from *this IC-bound*
have $Eq2$: $|IC\ l\ i\ (te\ m\ i) - IC\ m\ i\ (te\ m\ i)| \leq \delta S$
by (*simp add: max-def*)

hence $|IC\ m\ i\ (te\ m\ i) - IC\ l\ i\ (te\ m\ i)| \leq \delta S$
by (*simp add: AbsSym*)
from *this Eq1 corr-p corr-l corr-m pe-cond2*
have $|\vartheta\ p\ (i+1)\ m - \vartheta\ p\ (i+1)\ l| \leq ?Y$
by (*simp*)
thus *?thesis* **by** (*simp add: AbsSym*)
qed
qed

lemma four-one-ind-half:

assumes $ie1$: $\beta \leq rmin$
and $ie2$: $\mu \leq \delta S$
and $ie3$: $\gamma1\ \delta S \leq \delta S$
and $ind-hyp$: $okmaxsync\ i\ \delta S$
and $ie4$: $te\ q\ (i+1) \leq te\ p\ (i+1)$
and $corr-p$: $correct\ p\ (te\ p\ (i+1))$
and $corr-q$: $correct\ q\ (te\ p\ (i+1))$
shows $|IC\ p\ (i+1)\ (te\ p\ (i+1)) - IC\ q\ (i+1)\ (te\ p\ (i+1))| \leq \delta S$

proof-

let $?tpq = te\ p\ (i+1)$

let $?f = \lambda\ n.\ \vartheta\ q\ (i+1)\ n + (PC\ q\ (te\ p\ (i+1)) - PC\ q\ (te\ q\ (i+1)))$

let $?g = \vartheta\ p\ (i+1)$

from $ie4\ corr-q\ correct-closed$ **have** $corr-q-tq$: $correct\ q\ (te\ q\ (i+1))$
by *blast*

have $Eq-IC-cfn$: $|IC\ p\ (i+1)\ ?tpq - IC\ q\ (i+1)\ ?tpq| =$
 $|cfn\ q\ ?f - cfn\ p\ ?g|$

proof-

from $corr-p\ ICp-Suc$ **have** $Eq1$: $IC\ p\ (i+1)\ ?tpq = cfn\ p\ ?g$ **by** *simp*

from $ie4\ corr-p\ corr-q\ IC-trans-inv$

have $Eq2$: $IC\ q\ (i+1)\ ?tpq = cfn\ q\ ?f$ **by** *simp*

from $Eq1\ Eq2$ **show** *?thesis* **by** (*simp add: real-abs-def*)

qed

```

let ?ppred = λ l. correct l (te p (i+1))

let ?X = 2*ρ*β + 2*Λ
have ∀ l. ?ppred l → |?f l - ?g l| ≤ ?X
proof -
  {
    fix l
    assume ?ppred l
    from ie4 corr-p corr-q this pe-cond1
    have |?f l - ?g l| ≤ (2*ρ*β + 2*Λ)
      by(auto)
  }
  thus ?thesis by blast
qed
hence cond1: okRead2 ?f ?g ?X ?ppred
  by(simp add: okRead2-def)

let ?Y = 2*Λ + δS + 2*ρ*(rmax + β)

have ∀ l m. ?ppred l ∧ ?ppred m → |?f l - ?f m| ≤ ?Y
proof-
  {
    fix l m
    assume corr-l: ?ppred l
    assume corr-m: ?ppred m

    from corr-p corr-l beta-bound1 have tli-le-tp: te l i ≤ te p (i+1)
      by (simp add: le-diff-eq)
    from corr-p corr-m beta-bound1 have tmi-le-tp: te m i ≤ te p (i+1)
      by (simp add: le-diff-eq)

    let ?tlm = max (te l i) (te m i)

    from tli-le-tp tmi-le-tp have tlm-le-tp: ?tlm ≤ te p (i+1)
      by simp

    from ie4 corr-l correct-closed have corr-l-tq: correct l (te q (i+1))
      by blast
    from ie4 corr-m correct-closed have corr-m-tq: correct m (te q (i+1))
      by blast
    from tlm-le-tp corr-l correct-closed have corr-l-tlm: correct l ?tlm
      by blast
    from tlm-le-tp corr-m correct-closed have corr-m-tlm: correct m ?tlm
      by blast

    from ind-hyp corr-l-tlm corr-m-tlm
    have EqAbs1: |IC l i ?tlm - IC m i ?tlm| ≤ δS
      by(auto simp add: okmaxsync-def)

    have EqAbs3: |?f l - ?f m| = |ϑ q (i+1) l - ϑ q (i+1) m|
      by (simp add: real-abs-def)

    from EqAbs1 corr-q-tq corr-l-tq corr-m-tq theta-bound

```

have $|\vartheta q (i+1) l - \vartheta q (i+1) m| \leq ?Y$
by *simp*
from *this EqAbs3* **have** $|\?f l - \?f m| \leq ?Y$
by *simp*
}
thus *?thesis* **by** *simp*
qed
hence *cond2a: okRead1 ?f ?Y ?ppred* **by** (*simp add: okRead1-def*)

have $\forall l m. \?ppred l \wedge \?ppred m \longrightarrow |\?g l - \?g m| \leq ?Y$
proof-
{
fix *l m*
assume *corr-l: ?ppred l*
assume *corr-m: ?ppred m*

from *corr-p corr-l beta-bound1* **have** *tli-le-tp: te l i ≤ te p (i+1)*
by (*simp add: le-diff-eq*)
from *corr-p corr-m beta-bound1* **have** *tmi-le-tp: te m i ≤ te p (i+1)*
by (*simp add: le-diff-eq*)

let *?tln = max (te l i) (te m i)*
from *tli-le-tp tmi-le-tp* **have** *tlm-le-tp: ?tln ≤ te p (i+1)*
by *simp*

from *tlm-le-tp corr-l correct-closed* **have** *corr-l-tlm: correct l ?tln*
by *blast*
from *tlm-le-tp corr-m correct-closed* **have** *corr-m-tlm: correct m ?tln*
by *blast*

from *ind-hyp corr-l-tlm corr-m-tlm*
have *EqAbs1: |IC l i ?tln - IC m i ?tln| ≤ δS*
by(*auto simp add: okmaxsync-def*)

from *EqAbs1 corr-p corr-l corr-m theta-bound*
have $|\?g l - \?g m| \leq ?Y$ **by** *simp*
}
thus *?thesis* **by** *simp*
qed
hence *cond2b: okRead1 ?g ?Y ?ppred* **by** (*simp add: okRead1-def*)

from *correct-count* **have** $np - \text{maxfaults} \leq \text{count } \?ppred \ np$
by *simp*
from *this corr-p corr-q cond1 cond2a cond2b prec-enh*
have $|\text{cfn } q \?f - \text{cfn } p \?g| \leq \pi \?X \?Y$
by *blast*

from *ie3 this* **have** $|\text{cfn } q \?f - \text{cfn } p \?g| \leq \delta S$
by (*simp add: γ1-def*)

from *this Eq-IC-cfn* **show** *?thesis* **by** (*simp*)
qed

Theorem 4.1 in Shankar's paper.

```

theorem four-one:
  assumes ie1:  $\beta \leq rmin$ 
  and ie2:  $\mu \leq \delta S$ 
  and ie3:  $\gamma 1 \delta S \leq \delta S$ 
shows okmaxsync i  $\delta S$ 
proof(induct i)
  show okmaxsync 0  $\delta S$ 
  proof-
    {
      fix p q
      assume corr-p: correct p (max (te p 0) (te q 0))
      assume corr-q: correct q (max (te p 0) (te q 0))

      from corr-p synch0 have cp0: correct p 0 by simp
      from corr-q synch0 have cq0: correct q 0 by simp

      from synch0 cp0 cq0 IClock
      have IC-eq-PC:
        |IC p 0 (max (te p 0) (te q 0)) - IC q 0 (max (te p 0) (te q 0))|
        = |PC p 0 - PC q 0| (is ?T1 = ?T2)
        by(simp add: Adj-def)

      from ie2 init synch0 cp0 have range1:  $0 \leq PC p 0 \wedge PC p 0 \leq \delta S$ 
      by auto
      from ie2 init synch0 cq0 have range2:  $0 \leq PC q 0 \wedge PC q 0 \leq \delta S$ 
      by auto
      have ?T2  $\leq \delta S$ 
      proof cases
        assume A:  $PC p 0 < PC q 0$ 
        from A range1 range2 show ?thesis
          by(auto simp add: real-abs-def)
        next
        assume notA:  $\neg (PC p 0 < PC q 0)$ 
        from notA range1 range2 show ?thesis
          by(auto simp add: real-abs-def)
      qed
      from this IC-eq-PC have ?T1  $\leq \delta S$  by simp
    }
  thus ?thesis by (simp add: okmaxsync-def)
qed
next
fix i assume ind-hyp: okmaxsync i  $\delta S$ 
show okmaxsync (Suc i)  $\delta S$ 
proof-
  {
    fix p q
    assume corr-p: correct p (max (te p (i + 1)) (te q (i + 1)))
    assume corr-q: correct q (max (te p (i + 1)) (te q (i + 1)))
    let ?tp = te p (i + 1)
    let ?tq = te q (i + 1)
    let ?tpq = max ?tp ?tq

    have |IC p (i+1) ?tpq - IC q (i+1) ?tpq|  $\leq \delta S$  (is ?E1  $\leq \delta S$ )
  }

```

```

proof cases
  assume  $A: ?tq < ?tp$ 
  from  $A$  corr-p have  $cp1: \text{correct } p (te\ p\ (i+1))$ 
    by (simp add: max-def)
  from  $A$  corr-q have  $cq1: \text{correct } q (te\ p\ (i+1))$ 
    by (simp add: max-def)
  from  $A$ 
  have  $Eq1: ?E1 = |IC\ p\ (i+1)\ (te\ p\ (i+1)) - IC\ q\ (i+1)\ (te\ p\ (i+1))|$ 
    (is  $?E1 = ?E2$ )
    by (simp add: max-def)
  from  $A$   $cp1$   $cq1$  corr-p corr-q ind-hyp  $ie1$   $ie2$   $ie3$ 
    four-one-ind-half
  have  $?E2 \leq \delta S$  by (simp)
  from this  $Eq1$  show  $?thesis$  by simp
next
  assume  $notA: \neg (?tq < ?tp)$ 
  from this corr-p have  $cp2: \text{correct } p (te\ q\ (i+1))$ 
    by (simp add: max-def)
  from  $notA$  corr-q have  $cq2: \text{correct } q (te\ q\ (i+1))$ 
    by (simp add: max-def)
  from  $notA$ 
  have  $Eq2: ?E1 = |IC\ q\ (i+1)\ (te\ q\ (i+1)) - IC\ p\ (i+1)\ (te\ q\ (i+1))|$ 
    (is  $?E1 = ?E3$ )
    by (simp add: max-def AbsSym)
  from  $notA$  have  $?tp \leq ?tq$  by simp
  from this  $cp2$   $cq2$  ind-hyp  $ie1$   $ie2$   $ie3$  four-one-ind-half
  have  $?E3 \leq \delta S$ 
    by simp
  from this  $Eq2$  show  $?thesis$  by (simp)
qed
}
thus  $?thesis$  by (simp add: okmaxsync-def)
qed
qed

```

lemma *VC-cfn*:

```

  assumes corr-p:  $\text{correct } p (te\ p\ (i+1))$ 
  and  $ie: te\ p\ (i+1) < te\ p\ (i+2)$ 
shows  $VC\ p\ (te\ p\ (i+1)) = cfn\ p\ (\vartheta\ p\ (i+1))$ 
proof-
  from  $ie$  corr-p VClock have  $VC\ p\ (te\ p\ (i+1)) = IC\ p\ (i+1)\ (te\ p\ (i+1))$ 
    by simp
  moreover
  from corr-p IClock
  have  $IC\ p\ (i+1)\ (te\ p\ (i+1)) = PC\ p\ (te\ p\ (i+1)) + Adj\ p\ (i+1)$ 
    by blast
  moreover
  have  $PC\ p\ (te\ p\ (i+1)) + Adj\ p\ (i+1) = cfn\ p\ (\vartheta\ p\ (i+1))$ 
    by(simp add: Adj-def)
  ultimately show  $?thesis$  by simp
qed

```

Lemma for the inductive case in Theorem 4.2

lemma *four-two-ind*:
assumes *ie1*: $\beta \leq rmin$
and *ie2*: $\mu \leq \delta S$
and *ie3*: $\gamma^1 \delta S \leq \delta S$
and *ie4*: $\gamma^2 \delta S \leq \delta$
and *ie5*: $\gamma^3 \delta S \leq \delta$
and *ie6*: $te\ q\ (i+1) \leq te\ p\ (i+1)$
and *ind-hyp*: *okClocks* *p q i*
and *t-bound1*: $0 \leq t$
and *t-bound2*: $t < \max (te\ p\ (i+1)) (te\ q\ (i+1))$
and *t-bound3*: $\max (te\ p\ i) (te\ q\ i) \leq t$
and *tpq-bound*: $\max (te\ p\ i) (te\ q\ i) < \max (te\ p\ (i+1)) (te\ q\ (i+1))$
and *corr-p*: *correct* *p t*
and *corr-q*: *correct* *q t*
shows $|VC\ p\ t - VC\ q\ t| \leq \delta$
proof *cases*
assume *A*: $t < te\ q\ (i+1)$

let *?tpq* = $\max (te\ p\ i) (te\ q\ i)$

have *Eq1*: $te\ p\ i \leq t \wedge te\ q\ i \leq t$
proof *cases*
assume $te\ p\ i \leq te\ q\ i$
from *this* *t-bound3* **show** *?thesis* **by** (*simp add: max-def*)
next
assume $\neg (te\ p\ i \leq te\ q\ i)$
from *this* *t-bound3* **show** *?thesis* **by** (*simp add: max-def*)
qed

from *ie6* **have** *tp-max*: $\max (te\ p\ (i+1)) (te\ q\ (i+1)) = te\ p\ (i+1)$
by(*simp add: max-def*)
from *this* *t-bound2* **have** *Eq2*: $t < te\ p\ (i+1)$ **by** *simp*

from *VClock* *Eq1 Eq2 corr-p* **have** *Eq3*: $VC\ p\ t = IC\ p\ i\ t$ **by** *simp*

from *VClock* *Eq1 A corr-q* **have** *Eq4*: $VC\ q\ t = IC\ q\ i\ t$ **by** *simp*
from *Eq3 Eq4* **have** *Eq5*: $|VC\ p\ t - VC\ q\ t| = |IC\ p\ i\ t - IC\ q\ i\ t|$
by *simp*

from *t-bound3 corr-p corr-q correct-closed*
have *corr-tpq*: *correct* *p ?tpq* \wedge *correct* *q ?tpq*
by(*blast*)

from *t-bound3 IC-bd corr-p corr-q*
have *Eq6*: $|IC\ p\ i\ t - IC\ q\ i\ t| \leq |IC\ p\ i\ ?tpq - IC\ q\ i\ ?tpq|$
 $+ 2 * \rho * (t - ?tpq)$ (**is** *?E1* \leq *?E2*)
by(*blast*)

from *ie1 ie2 ie3 four-one* **have** *okmaxsync* *i* δS **by** *simp*

from *this* *corr-tpq* **have** $|IC\ p\ i\ ?tpq - IC\ q\ i\ ?tpq| \leq \delta S$
by(*simp add: okmaxsync-def*)

from *Eq6 this* **have** *Eq7: ?E1* $\leq \delta S + 2 * \rho * (t - ?tpq)$ **by** *simp*

from *corr-p Eq2 rts0* **have** $t - te\ p\ i \leq rmax$ **by** *simp*

from *this* **have** $t - ?tpq \leq rmax$ **by** (*simp add: max-def*)

from *this constants-ax* **have** $2 * \rho * (t - ?tpq) \leq 2 * \rho * rmax$

by (*simp add: real-mult-le-cancel-iff1*)

hence $\delta S + 2 * \rho * (t - ?tpq) \leq \delta S + 2 * \rho * rmax$

by *simp*

from *this Eq7* **have** *?E1* $\leq \delta S + 2 * \rho * rmax$ **by** *simp*

from *this Eq5 ie4* **show** *?thesis* **by** (*simp add: \gamma2-def*)

next

assume $\neg (t < te\ q\ (i+1))$

hence *B: te q (i+1) ≤ t* **by** *simp*

from *ie6 t-bound2*

have *tp-max: max (te p (i+1)) (te q (i+1)) = te p (i+1)*

by (*simp add: max-def*)

have $te\ p\ i \leq \max (te\ p\ i) (te\ q\ i)$

by (*simp add: max-def*)

from *this t-bound3* **have** *tp-bound1: te p i ≤ t* **by** *simp*

from *tp-max t-bound2* **have** *tp-bound2: t < te p (i+1)* **by** *simp*

have *tq-bound1: t < te q (i+2)*

proof (*rule ccontr*)

assume $\neg (t < te\ q\ (i+2))$

hence *C: te q (i+2) ≤ t* **by** *simp*

from *C corr-q correct-closed*

have *corr-q-t2: correct q (te q (i+2))* **by** *blast*

have $te\ q\ (i+1) + \beta \leq t$

proof–

from *corr-q-t2 rts1d* **have** $rmin \leq te\ q\ (i+2) - te\ q\ (i+1)$

by *simp*

from *this ie1* **have** $\beta \leq te\ q\ (i+2) - te\ q\ (i+1)$

by *simp*

hence $te\ q\ (i+1) + \beta \leq te\ q\ (i+2)$ **by** *simp*

from *this C* **show** *?thesis* **by** *simp*

qed

from *this corr-p corr-q rts2a* **have** $te\ p\ (i+1) \leq t$

by *blast*

hence $\neg (t < te\ p\ (i+1))$ **by** *simp*

from *this tp-bound2* **show** *False* **by** *simp*

qed

from *tq-bound1 B* **have** *tq-bound2: te q (i+1) < te q (i+2)* **by** *simp*

from *B tp-bound2* **have** *tq-bound3: te q (i+1) < te p (i+1)*

by *simp*

from *B corr-p correct-closed*

have *corr-p-tq1: correct p (te q (i+1))* **by** *blast*

from B *correct-closed corr-q*
have corr-q-tq1 : $\text{correct } q (te \ q \ (i+1))$ **by** *blast*

from corr-p-tq1 corr-q-tq1 *beta-bound1*
have tq-bound4 : $te \ p \ i \leq te \ q \ (i+1)$
by (*simp add: le-diff-eq*)

from tq-bound1 *VClock B corr-q*
have Eq1 : $VC \ q \ t = IC \ q \ (i+1) \ t$ **by** *simp*

from *VClock tp-bound1 tp-bound2 corr-p*
have Eq2 : $VC \ p \ t = IC \ p \ i \ t$ **by** *simp*

from Eq1 Eq2 **have** Eq3 : $|VC \ p \ t - VC \ q \ t| = |IC \ p \ i \ t - IC \ q \ (i+1) \ t|$
by *simp*

from B *corr-p corr-q IC-bd*
have $|IC \ p \ i \ t - IC \ q \ (i+1) \ t| \leq$
 $|IC \ p \ i \ (te \ q \ (i+1)) - IC \ q \ (i+1) \ (te \ q \ (i+1))| + 2 * \rho * (t - te \ q \ (i+1))$
by *simp*

from *this Eq3*
have *VC-split*: $|VC \ p \ t - VC \ q \ t| \leq$
 $|IC \ p \ i \ (te \ q \ (i+1)) - IC \ q \ (i+1) \ (te \ q \ (i+1))| + 2 * \rho * (t - te \ q \ (i+1))$
by *simp*

from tq-bound2 *VClock corr-q-tq1*
have Eq4 : $VC \ q \ (te \ q \ (i+1)) = IC \ q \ (i+1) \ (te \ q \ (i+1))$ **by** *simp*

from *this tq-bound2 VC-cfn corr-q-tq1*
have Eq5 : $IC \ q \ (i+1) \ (te \ q \ (i+1)) = \text{cfn } q \ (\vartheta \ q \ (i+1))$ **by** *simp*

hence *IC-eq-cfn*: $IC \ p \ i \ (te \ q \ (i+1)) - IC \ q \ (i+1) \ (te \ q \ (i+1)) =$
 $IC \ p \ i \ (te \ q \ (i+1)) - \text{cfn } q \ (\vartheta \ q \ (i+1))$
(is ?E1 = ?E2)
by *simp*

let $?f = \vartheta \ q \ (i+1)$
let $?ppred = \lambda \ l. \text{correct } l \ (te \ q \ (i+1))$
let $?X = 2 * \Lambda + \delta S + 2 * \rho * (rmax + \beta)$

have $\forall \ l \ m. ?ppred \ l \wedge ?ppred \ m \longrightarrow |\vartheta \ q \ (i+1) \ l - \vartheta \ q \ (i+1) \ m| \leq ?X$
proof–
 $\{$
fix $l :: \text{process}$
fix $m :: \text{process}$
assume $\text{corr-l}: ?ppred \ l$
assume $\text{corr-m}: ?ppred \ m$

let $?tlm = \max (te \ l \ i) (te \ m \ i)$
have $\text{tlm-bound}: ?tlm \leq te \ q \ (i+1)$
proof–

from *corr-l corr-q-tq1 beta-bound1* **have** $te\ l\ i \leq te\ q\ (i+1)$
by (*simp add: le-diff-eq*)
moreover
from *corr-m corr-q-tq1 beta-bound1* **have** $te\ m\ i \leq te\ q\ (i+1)$
by (*simp add: le-diff-eq*)
ultimately show *?thesis* **by** *simp*
qed

from *tlm-bound corr-l corr-m correct-closed*
have *corr-tlm: correct l ?tlm \wedge correct m ?tlm*
by *blast*

have $|IC\ l\ i\ ?tlm - IC\ m\ i\ ?tlm| \leq \delta S$
proof–
from *ie1 ie2 ie3 four-one* **have** *okmaxsync i δS*
by *simp*
from *this corr-tlm* **show** *?thesis* **by**(*simp add: okmaxsync-def*)
qed

from *this corr-l corr-m corr-q-tq1 theta-bound*
have $|\vartheta\ q\ (i+1)\ l - \vartheta\ q\ (i+1)\ m| \leq ?X$ **by** *simp*
}
thus *?thesis* **by** *blast*
qed

hence *readOK: okRead1 ($\vartheta\ q\ (i+1)$) ?X ?ppred*
by(*simp add: okRead1-def*)

let $?E3 = cfn\ q\ (\vartheta\ q\ (i+1)) - \vartheta\ q\ (i+1)\ p$
let $?E4 = \vartheta\ q\ (i+1)\ p - IC\ p\ i\ (te\ q\ (i+1))$

have $|\?E2| = |\?E3 + ?E4|$ **by** (*simp add: real-abs-def*)
hence *Eq8: $|\?E2| \leq |\?E3| + |\?E4|$* **by** (*simp add: real-abs-def*)

from *correct-count* **have** *ppredOK: np - maxfaults \leq count ?ppred np*
by *simp*
from *readOK ppredOK corr-p-tq1 corr-q-tq1 acc-prsv*
have $|\?E3| \leq \alpha\ ?X$
by *blast*
from *this Eq8* **have** *Eq9: $|\?E2| \leq \alpha\ ?X + |\?E4|$* **by** *simp*

from *corr-p-tq1 corr-q-tq1 readerror*
have $|\?E4| \leq \Lambda$ **by** *simp*

from *this Eq9* **have** *Eq10: $|\?E2| \leq \alpha\ ?X + \Lambda$* **by** *simp*

from *this VC-split IC-eq-cfn*
have *almost-right:*
 $|VC\ p\ t - VC\ q\ t| \leq$
 $\alpha\ ?X + \Lambda + 2*\rho*(t - te\ q\ (i+1))$
by *simp*

have $t - te\ q\ (i+1) \leq \beta$
proof (*rule ccontr*)

```

assume  $\neg (t - te\ q\ (i+1) \leq \beta)$ 
hence  $te\ q\ (i+1) + \beta \leq t$  by simp
from this corr-p corr-q rts2a have  $te\ p\ (i+1) \leq t$ 
  by auto
hence  $\neg (t < te\ p\ (i+1))$  by simp
from this tp-bound2 show False
  by simp
qed

```

```

from this constants-ax
have  $\alpha\ ?X + \Lambda + 2*\rho*(t - te\ q\ (i+1))$ 
   $\leq \alpha\ ?X + \Lambda + 2*\rho*\beta$ 
  by (simp)

```

```

from this almost-right
have  $|VC\ p\ t - VC\ q\ t| \leq \alpha\ ?X + \Lambda + 2*\rho*\beta$ 
  by arith

```

```

from this ie5 show ?thesis by (simp add: \gamma3-def)
qed

```

Theorem 4.2 in Shankar's paper.

theorem *four-two:*

```

assumes ie1: \beta \leq rmin
and ie2: \mu \leq \delta S
and ie3: \gamma1 \delta S \leq \delta S
and ie4: \gamma2 \delta S \leq \delta
and ie5: \gamma3 \delta S \leq \delta
shows okClocks p q i
proof (induct i)
  show okClocks p q 0
  proof–
  {
    fix t :: time
    assume t-bound1: 0 \leq t
    assume t-bound2: t < max (te p 0) (te q 0)
    assume corr-p: correct p t
    assume corr-q: correct q t
    from t-bound2 synch0 have  $t < 0$ 
      by(simp add: max-def)
    from this t-bound1 have False by simp
    hence  $|VC\ p\ t - VC\ q\ t| \leq \delta$  by simp
  }
  thus ?thesis by (simp add: okClocks-def)

```

qed

next

```

fix i::nat assume ind-hyp: okClocks p q i
show okClocks p q (Suc i)
proof –
  {
    fix t :: time
    assume t-bound1: 0 \leq t
    assume t-bound2: t < max (te p (i+1)) (te q (i+1))

```

```

assume corr-p: correct p t
assume corr-q: correct q t

let ?tpq1 = max (te p i) (te q i)
let ?tpq2 = max (te p (i+1)) (te q (i+1))

have  $|VC\ p\ t - VC\ q\ t| \leq \delta$ 
proof cases
  assume tpq-bound: ?tpq1 < ?tpq2
  show ?thesis
  proof cases
    assume  $t < ?tpq1$ 
    from t-bound1 this corr-p corr-q ind-hyp
    show ?thesis by(simp add: okClocks-def)
  next
    assume  $\neg (t < ?tpq1)$ 
    hence tpq-le-t:  $?tpq1 \leq t$  by arith

    show ?thesis
    proof cases
      assume A:  $te\ q\ (i+1) \leq te\ p\ (i+1)$ 

      from this tpq-le-t tpq-bound ie1 ie2 ie3 ie4 ie5
      ind-hyp t-bound1 t-bound2
      corr-p corr-q tpq-bound four-two-ind
      show ?thesis by(simp)
    next
      assume  $\neg (te\ q\ (i+1) \leq te\ p\ (i+1))$ 
      hence B:  $te\ p\ (i+1) \leq te\ q\ (i+1)$  by simp

      from ind-hyp okClocks-sym have ind-hyp1: okClocks q p i
      by blast

      have maxsym1:  $max (te\ p\ (i+1)) (te\ q\ (i+1)) = max (te\ q\ (i+1)) (te\ p\ (i+1))$ 
      by (simp add: max-def)
      have maxsym2:  $max (te\ p\ i) (te\ q\ i) = max (te\ q\ i) (te\ p\ i)$ 
      by (simp add: max-def)

      from maxsym1 t-bound2
      have t-bound21:  $t < max (te\ q\ (i+1)) (te\ p\ (i+1))$ 
      by simp

      from maxsym1 maxsym2 tpq-bound
      have tpq-bound1:  $max (te\ q\ i) (te\ p\ i) < max (te\ q\ (i+1)) (te\ p\ (i+1))$ 
      by simp
      from maxsym2 tpq-le-t
      have tpq-le-t1:  $max (te\ q\ i) (te\ p\ i) \leq t$  by simp

      from B tpq-le-t1 tpq-bound1 ie1 ie2 ie3 ie4 ie5
      ind-hyp1 t-bound1 t-bound21
      corr-p corr-q tpq-bound four-two-ind
      have  $|VC\ q\ t - VC\ p\ t| \leq \delta$  by(simp)
      thus ?thesis by (simp add: AbsSym)

```

```

    qed
  qed
next
  assume  $\neg (?tpq1 < ?tpq2)$ 
  hence  $?tpq2 \leq ?tpq1$  by arith
  from t-bound2 this have  $t < ?tpq1$  by arith
  from t-bound1 this corr-p corr-q ind-hyp
  show ?thesis by (simp add: okClocks-def)
qed
}
thus ?thesis by (simp add: okClocks-def)
qed
qed

```

The main theorem: all correct clocks are synchronized within the bound delta.

theorem *agreement*:

```

  assumes ie1:  $\beta \leq rmin$ 
  and ie2:  $\mu \leq \delta S$ 
  and ie3:  $\gamma1 \delta S \leq \delta S$ 
  and ie4:  $\gamma2 \delta S \leq \delta$ 
  and ie5:  $\gamma3 \delta S \leq \delta$ 
  and ie6:  $0 \leq t$ 
  and cpq: correct p t  $\wedge$  correct q t
shows  $|VC\ p\ t - VC\ q\ t| \leq \delta$ 

```

proof–

```

  from ie6 cpq event-bound have  $\exists i :: nat. t < \max (te\ p\ i) (te\ q\ i)$ 
  by simp
  from this obtain i :: nat where t-bound:  $t < \max (te\ p\ i) (te\ q\ i) ..$ 

```

```

  from t-bound ie1 ie2 ie3 ie4 ie5 four-two have okClocks p q i
  by simp

```

```

  from ie6 this t-bound cpq show ?thesis
  by (simp add: okClocks-def)

```

qed

end

References

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