

# Secure information flow and program logics — Isabelle/HOL sources

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## Abstract

We present interpretations of type systems for secure information flow in Hoare logic, complementing previous encodings in relational program logics. We first treat the imperative language **IMP**, extended by a simple procedure call mechanism. For this language we consider base-line non-interference in the style of Volpano et al. [8] and the flow-sensitive type system by Hunt and Sands [4]. In both cases, we show how typing derivations may be used to automatically generate proofs in the program logic that certify the absence of illicit flows. We then add instructions for object creation and manipulation, and derive appropriate proof rules for base-line non-interference. As a consequence of our work, standard verification technology may be used for verifying that a concrete program satisfies the non-interference property.

The present proof development represents an update of the formalisation underlying our paper [2] and is intended to resolve any ambiguities that may be present in the paper.

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**theory IMP imports Main begin**

## **1 The language IMP**

In this section we define a simple imperative programming language. Syntax and operational semantics are as in [9], except that we enrich the language with a single unnamed, parameterless procedure. Both, this section and the following one merely set the basis for the development described in the later sections and largely follow the approach to formalize program logics advocated by Kleymann, Nipkow, and others - see for example [5, 6, 7].

## 1.1 Syntax

We start from unspecified categories of program variables and values.

**typedecl** *Var*

**typedecl** *Val*

Arithmetic expressions are inductively built up from variables, values, and binary operators which are modeled as meta-logical functions over values. Similarly, boolean expressions are built up from arithmetic expressions using binary boolean operators which are modeled as functions of the ambient logic HOL.

**datatype** *Expr* =

*varE* *Var*

| *valE* *Val*

| *opE* *Val*  $\Rightarrow$  *Val*  $\Rightarrow$  *Val Expr Expr*

**datatype** *BExpr* = *compB* *Val*  $\Rightarrow$  *Val*  $\Rightarrow$  *bool Expr Expr*

Commands are the usual ones for an imperative language, plus the command *Call* which stands for the invocation of a single (unnamed, parameterless) procedure.

**datatype** *IMP* =

*Skip*

| *Assign* *Var Expr*

| *Comp* *IMP IMP*

| *While* *BExpr IMP*

| *Iff* *BExpr IMP IMP*

| *Call*

The body of this procedure is identified by the following constant.

**consts** *body* :: *IMP*

## 1.2 Dynamic semantics

States are given by stores - in our case, HOL functions mapping program variables to values.

**types** *State* = *Var*  $\Rightarrow$  *Val*

**definition** *update* :: *State*  $\Rightarrow$  *Var*  $\Rightarrow$  *Val*  $\Rightarrow$  *State*

**where** *update* *s* *x* *v* = ( $\lambda$  *y* . if *x=y* then *v* else *s* *y*)

The evaluation of expressions is defined inductively, as standard.

**primrec** *evalE*::*Expr*  $\Rightarrow$  *State*  $\Rightarrow$  *Val*

**where**

*evalE* (*varE* *x*) *s* = *s* *x* |

*evalE* (*valE* *v*) *s* = *v* |

*evalE* (*opE* *f* *e1* *e2*) *s* = *f* (*evalE* *e1* *s*) (*evalE* *e2* *s*)

**primrec**  $evalB :: BExpr \Rightarrow State \Rightarrow bool$   
**where**  
 $evalB (compB f e1 e2) s = f (evalE e1 s) (evalE e2 s)$

The operational semantics is a standard big-step relation, with a height index that facilitates the Kleymann-Nipkow-style [5, 6] soundness proof of the program logic.

**inductive-set**  $Semn :: (State \times IMP \times nat \times State) set$  **where**  
 $SemSkip: (s, Skip, 1, s) : Semn$

|  $SemAssign:$   
 $\llbracket t = update\ s\ x\ (evalE\ e\ s) \rrbracket \Longrightarrow (s, Assign\ x\ e, 1, t) : Semn$

|  $SemComp:$   
 $\llbracket (s, c1, n, r) : Semn; (r, c2, m, t) : Semn; k = (max\ n\ m) + 1 \rrbracket$   
 $\Longrightarrow (s, Comp\ c1\ c2, k, t) : Semn$

|  $SemWhileT:$   
 $\llbracket evalB\ b\ s; (s, c, n, r) : Semn; (r, While\ b\ c, m, t) : Semn;$   
 $k = ((max\ n\ m) + 1) \rrbracket$   
 $\Longrightarrow (s, While\ b\ c, k, t) : Semn$

|  $SemWhileF:$   $\llbracket \neg (evalB\ b\ s); t = s \rrbracket \Longrightarrow (s, While\ b\ c, 1, t) : Semn$

|  $SemTrue:$   
 $\llbracket evalB\ b\ s; (s, c1, n, t) : Semn \rrbracket \Longrightarrow (s, Iff\ b\ c1\ c2, n + 1, t) : Semn$

|  $SemFalse:$   
 $\llbracket \neg (evalB\ b\ s); (s, c2, n, t) : Semn \rrbracket \Longrightarrow (s, Iff\ b\ c1\ c2, n + 1, t) : Semn$

|  $SemCall:$   $(s, body, n, t) : Semn \Longrightarrow (s, Call, n + 1, t) : Semn$

### abbreviation

$SemN :: [State, IMP, nat, State] \Rightarrow bool$   $(-, - \rightarrow_n -)$   
**where**  
 $s, c \rightarrow_n t == (s, c, n, t) : Semn$

Often, the height index does not matter, so we define a notion hiding it.

**definition**  $Sem :: [State, IMP, State] \Rightarrow bool$   $(-, - \Downarrow - 1000)$   
**where**  $s, c \Downarrow t = (\exists n. s, c \rightarrow_n t)$

Inductive elimination rules for the (indexed) dynamic semantics:

**inductive-cases**  $Sem\text{-eval-cases}:$

$s, Skip \rightarrow_n t$   
 $s, (Assign\ x\ e) \rightarrow_n t$   
 $s, (Comp\ c1\ c2) \rightarrow_n t$   
 $s, (While\ b\ c) \rightarrow_n t$   
 $s, (Iff\ b\ c1\ c2) \rightarrow_n t$

$s, Call \rightarrow_n t$   
 $\langle proof \rangle$

An induction on  $c$  shows that no derivations of height 0 exist.

**lemma** *Sem-no-zero-height-derivus*:  $(s, c \rightarrow_0 t) ==> False \langle proof \rangle \langle proof \rangle$

The proof of determinism is by induction on the (indexed) operational semantics.

**lemma** *SemDeterm*:  $\llbracket s, c \Downarrow t; s, c \Downarrow r \rrbracket \implies r=t \langle proof \rangle$

End of theory IMP

**end**

**theory** *VDM* **imports** *IMP* **begin**

## 2 Program logic

The program logic is a partial correctness logic in (precondition-less) VDM style. This means that assertions are binary predicates over states and relate the initial and final states of a terminating execution.

### 2.1 Assertions and their semantic validity

Assertions are binary predicates over states, i.e. are of type

**types** *VDMAssn* = *State*  $\Rightarrow$  *State*  $\Rightarrow$  *bool*

Command  $c$  satisfies assertion  $A$  if all (terminating) operational behaviours are covered by the assertion.

**definition** *VDM-valid* :: *IMP*  $\Rightarrow$  *VDMAssn*  $\Rightarrow$  *bool*  
 $(\models - : - [100,100] 100)$

**where**  $\models c : A = (\forall s t . (s, c \Downarrow t) \longrightarrow A s t)$

A variation of this property for the height-indexed operational semantics,...

**definition** *VDM-validn* :: *nat*  $\Rightarrow$  *IMP*  $\Rightarrow$  *VDMAssn*  $\Rightarrow$  *bool*  
 $(\models_n - : - [100,100,100] 100)$

**where**  $\models_n c : A = (\forall m . m \leq n \longrightarrow (\forall s t . (s, c \rightarrow_m t) \longrightarrow A s t))$

... plus the obvious relationships.

**lemma** *VDM-valid-validn*:  $\models c:A \implies \models_n c:A \langle proof \rangle$

**lemma** *VDM-validn-valid*:  $(\forall n . \models_n c:A) \implies \models c:A \langle proof \rangle$

**lemma** *VDM-lowerm*:  $\llbracket \models_n c:A; m \leq n \rrbracket \implies \models_m c:A \langle proof \rangle$

Proof contexts are simply sets of assertions – each entry represents an assumption for the unnamed procedure. In particular, a context is valid if each entry is satisfied by the method call instruction.

**definition** *Ctxt-valid* ::  $VDMAssn\ set \Rightarrow bool$  ( $\models - [100] 100$ )  
**where**  $\models G = (\forall A . A \in G \longrightarrow (\models Call : A))$

Again, a relativised sibling ...

**definition** *Ctxt-validn* ::  $nat \Rightarrow (VDMAssn\ set) \Rightarrow bool$   
 $(\models_n - [100,100] 100)$   
**where**  $\models_n G = (\forall m . m \leq n \longrightarrow (\forall A . A \in G \longrightarrow (\models_m Call : A)))$

satisfies the obvious properties.

**lemma** *Ctxt-valid-validn*:  $\models G \Longrightarrow \models_n G \langle proof \rangle$   
**lemma** *Ctxt-validn-valid*:  $(\forall n . \models_n G) \Longrightarrow \models G \langle proof \rangle$   
**lemma** *Ctxt-lowerm*:  $\llbracket \models_n G; m < n \rrbracket \Longrightarrow \models_m G \langle proof \rangle$

A judgement is valid if the validity of the context implies that of the command-assertion pair.

**definition** *valid* ::  $(VDMAssn\ set) \Rightarrow IMP \Rightarrow VDMAssn \Rightarrow bool$   
 $(- \models - : - [100,100,100] 100)$   
**where**  $G \models c : A = (\models G \longrightarrow \models c : A)$

And, again, a related notion of judgement validity.

**definition** *validn* ::  
 $(VDMAssn\ set) \Rightarrow nat \Rightarrow IMP \Rightarrow VDMAssn \Rightarrow bool$   
 $(- \models_n - : - [100,100,100,100] 100)$   
**where**  $G \models_n c : A = (\models_n G \longrightarrow \models_n c : A)$

**lemma** *validn-valid*:  $(\forall n . G \models_n c : A) \Longrightarrow G \models c : A \langle proof \rangle$   
**lemma** *ctxt-consn*:  $\llbracket \models_n G; \models_n Call:A \rrbracket \Longrightarrow \models_n (\{A\} \cup G) \langle proof \rangle$

## 2.2 Proof system

### inductive-set

*VDM-proof* ::  $(VDMAssn\ set \times IMP \times VDMAssn)\ set$   
**where**

*VDMSkip*:  $(G, Skip, \lambda s\ t . t=s) : VDM-proof$

| *VDMAssign*:  
 $(G, Assign\ x\ e, \lambda s\ t . t = (update\ s\ x\ (evalE\ e\ s))) : VDM-proof$

| *VDMComp*:  
 $\llbracket (G, c1, A1) : VDM-proof; (G, c2, A2) : VDM-proof \rrbracket \Longrightarrow$   
 $(G, Comp\ c1\ c2, \lambda s\ t . \exists r . A1\ s\ r \wedge A2\ r\ t) : VDM-proof$

| *VDMIff*:  
 $\llbracket (G, c1, A) : VDM-proof; (G, c2, B) : VDM-proof \rrbracket \Longrightarrow$   
 $(G, Iff\ b\ c1\ c2, \lambda s\ t . (((evalB\ b\ s) \longrightarrow A\ s\ t) \wedge$   
 $((\neg (evalB\ b\ s)) \longrightarrow B\ s\ t))) : VDM-proof$

| *VDMWhile*:  

$$\llbracket (G, c, B):VDM\text{-proof}; \forall s. (\neg \text{eval}B\ b\ s) \longrightarrow A\ s\ s;$$

$$\forall s\ r\ t. \text{eval}B\ b\ s \longrightarrow B\ s\ r \longrightarrow A\ r\ t \longrightarrow A\ s\ t \rrbracket \Longrightarrow$$

$$(G, \text{While}\ b\ c, \lambda s\ t. A\ s\ t \wedge \neg (\text{eval}B\ b\ t)) : VDM\text{-proof}$$

| *VDMCall*:  

$$(\{A\} \cup G, \text{body}, A):VDM\text{-proof} \Longrightarrow (G, \text{Call}, A):VDM\text{-proof}$$

| *VDMAx*:  $A \in G \Longrightarrow (G, \text{Call}, A):VDM\text{-proof}$

| *VDMConseq*:  

$$\llbracket (G, c, A):VDM\text{-proof}; \forall s\ t. A\ s\ t \longrightarrow B\ s\ t \rrbracket \Longrightarrow$$

$$(G, c, B):VDM\text{-proof}$$

### abbreviation

$VDM\text{-deriv} :: [VDM\text{Assn}\ \text{set}, IMP, VDM\text{Assn}] \Rightarrow \text{bool}$   
 $(- \triangleright - : - [100,100,100] 100)$

where  $G \triangleright c : A == (G, c, A) \in VDM\text{-proof}$

The while-rule is in fact inter-derivable with the following rule.

**lemma** *Hoare-While*:

$$G \triangleright c : (\lambda s\ s'. \forall r. \text{eval}B\ b\ s \longrightarrow I\ s\ r \longrightarrow I\ s'\ r) \Longrightarrow$$

$$G \triangleright \text{While}\ b\ c : (\lambda s\ s'. \forall r. I\ s\ r \longrightarrow (I\ s'\ r \wedge \neg \text{eval}B\ b\ s'))$$

$$\langle \text{proof} \rangle$$

Here's the proof in the opposite direction.

**lemma** *VDMWhile-derivable*:

$$\llbracket G \triangleright c : B; \forall s. (\neg \text{eval}B\ b\ s) \longrightarrow A\ s\ s;$$

$$\forall s\ r\ t. \text{eval}B\ b\ s \longrightarrow B\ s\ r \longrightarrow A\ r\ t \longrightarrow A\ s\ t \rrbracket$$

$$\Longrightarrow G \triangleright (\text{While}\ b\ c) : (\lambda s\ t. A\ s\ t \wedge \neg (\text{eval}B\ b\ t))$$

$$\langle \text{proof} \rangle$$

## 2.3 Soundness

$\langle \text{proof} \rangle \langle \text{proof} \rangle$

An auxiliary lemma stating the soundness of the while rule. Its proof is by induction on  $n$ .

**lemma** *SoundWhile*[*rule-format*]:

$$(\forall m. G \models_m c : B) \longrightarrow (\forall s. (\neg \text{eval}B\ b\ s) \longrightarrow A\ s\ s) \longrightarrow$$

$$(\forall s. \text{eval}B\ b\ s \longrightarrow (\forall r. B\ s\ r \longrightarrow (\forall t. A\ r\ t \longrightarrow A\ s\ t))) \longrightarrow$$

$$G \models_n (\text{While}\ b\ c) : (\lambda s\ t. A\ s\ t \wedge \neg \text{eval}B\ b\ t) \langle \text{proof} \rangle$$

Similarly, an auxiliary lemma for procedure invocations. Again, the proof proceeds by induction on  $n$ .

**lemma** *SoundCall*[*rule-format*]:

$$\llbracket \forall n. \models_n (\{A\} \cup G) \longrightarrow \models_n \text{body} : A \rrbracket \Longrightarrow \models_n G \longrightarrow \models_n \text{Call} : A \langle \text{proof} \rangle$$

The heart of the soundness proof is the following lemma which is proven by induction on the judgement  $G \triangleright c : A$ .

**lemma** *VDM-Sound-n*:  $G \triangleright c : A \implies (\forall n . G \models_n c:A)\langle proof \rangle$

Combining this result with lemma *validn-valid*, we obtain soundness in contexts,...

**theorem** *VDM-Sound*:  $G \triangleright c : A \implies G \models c:A\langle proof \rangle$

...and consequently soundness w.r.t. empty contexts.

**lemma** *VDM-Sound-emptyCtxt*:  $\{\} \triangleright c : A \implies \models c : A\langle proof \rangle$

## 2.4 Admissible rules

A weakening rule and some cut rules are easily derived.

**lemma** *WEAK* [*rule-format*]:

$G \triangleright c : A \implies (\forall H . G \subseteq H \longrightarrow H \triangleright c : A)\langle proof \rangle\langle proof \rangle$

**lemma** *CutAux*:

$\llbracket H \triangleright c : A ; H = \text{insert } P D ; G \triangleright \text{Call } :P ; G \subseteq D \rrbracket \implies D \triangleright c:A\langle proof \rangle$

**lemma** *Cut*:  $\llbracket G \triangleright \text{Call} : P ; (\text{insert } P G) \triangleright c : A \rrbracket \implies G \triangleright c : A\langle proof \rangle$

We call context  $G$  verified if all entries are justified by derivations for the procedure body.

**definition** *verified*:  $VDMAssn \text{ set} \Rightarrow \text{bool}$

**where** *verified*  $G = (\forall A . A:G \longrightarrow G \triangleright \text{body} : A)$

The property is preserved by sub-contexts

**lemma** *verified-preserved*:  $\llbracket \text{verified } G ; A:G \rrbracket \implies \text{verified } (G - \{A\})\langle proof \rangle\langle proof \rangle\langle proof \rangle$

The *Mutrec* rule allows us to eliminate verified (finite) contexts. Its proof proceeds by induction on  $n$ .

**theorem** *Mutrec*:

$\llbracket \text{finite } G ; \text{card } G = n ; \text{verified } G ; A : G \rrbracket \implies \{\} \triangleright \text{Call}:A\langle proof \rangle$

In particular, *Mutrec* may be used to show that verified finite contexts are valid.

**lemma** *Ctxt-verified-valid*:  $\llbracket \text{verified } G ; \text{finite } G \rrbracket \implies \models G\langle proof \rangle$

## 2.5 Completeness

Strongest specifications, given precisely by the operational behaviour.

**definition** *SSpec*:  $IMP \Rightarrow VDMAssn$

**where** *SSpec*  $c \ s \ t = s, c \Downarrow t$

Strongest specifications are valid ...

**lemma** *SSpec-valid*:  $\models c : (SSpec \ c)\langle proof \rangle$

and imply any other valid assertion for the same program (hence their name).

**lemma** *SSpec-strong*:  $\models c : A \implies \forall s \ t . SSPEC \ c \ s \ t \longrightarrow A \ s \ t\langle proof \rangle$

By induction on  $c$  we show the following.

**lemma** *SSpec-derivable*:  $G \triangleright Call : SSpec Call \implies G \triangleright c : SSpec c \langle proof \rangle$

The (singleton) strong context contains the strongest specification of the procedure.

**definition** *StrongG* :: *VDMAssn set*  
**where** *StrongG* = {*SSpec Call*}

By construction, the strongest specification of the procedure's body can be verified with respect to this context.

**lemma** *StrongG-Body*:  $StrongG \triangleright body : SSpec Call \langle proof \rangle$

Thus, the strong context is verified.

**lemma** *StrongG-verified*:  $verified StrongG \langle proof \rangle$

Using this result and the rules *Cut* and *Mutrec*, we show that arbitrary commands satisfy their strongest specification with respect to the empty context.

**lemma** *SSpec-derivable-empty*:  $\{\} \triangleright c : SSpec c \langle proof \rangle$

From this, we easily obtain (relative) completeness.

**theorem** *VDM-Complete*:  $\models c : A \implies \{\} \triangleright c : A \langle proof \rangle$

Finally, it is easy to show that valid contexts are verified.

**lemma** *Ctxt-valid-verified*:  $\models G \implies verified G \langle proof \rangle$

End of theory VDM

**end**

**theory** *VS imports VDM begin*

### 3 Base-line noninterference

We now show how to interpret the type system of Volpano, Smith and Irvine [8], as described in Section 3 of our paper [2].

#### 3.1 Basic definitions

Multi-level security being treated in Section 5, we restrict our attention in the present section to the two-point security lattice.

**datatype** *TP* = *low* | *high*

A global context assigns a security type to each program variable.

**consts** *CONTEXT* :: *Var*  $\Rightarrow$  *TP*

Next, we define when two states are considered (low) equivalent.

**definition** *twiddle*:: $State \Rightarrow State \Rightarrow bool$  ( $- \approx - [100,100] 100$ )  
**where**  $s \approx ss = (\forall x. CONTEXT\ x = low \longrightarrow s\ x = ss\ x)$

A command  $c$  is *secure* if the low equivalence of any two initial states entails the equivalence of the corresponding final states.

**definition** *secure*:: $IMP \Rightarrow bool$   
**where**  $secure\ c = (\forall s\ t\ ss\ tt. s \approx t \longrightarrow (s, c \Downarrow ss) \longrightarrow (t, c \Downarrow tt) \longrightarrow ss \approx tt)$

Here is the definition of the assertion transformer that is called *Sec* in the paper ...

**definition** *Sec* ::  $((State \times State) \Rightarrow bool) \Rightarrow VDMA_{ssn}$   
**where**  $Sec\ \Phi\ s\ t = ((\forall r. s \approx r \longrightarrow \Phi(t, r)) \wedge (\forall r. \Phi(r, s) \longrightarrow r \approx t))$

... and the proofs of two directions of its characteristic property, Proposition 1.

**lemma** *Prop1A*:  $\models c : (Sec\ \Phi) \Longrightarrow secure\ c \langle proof \rangle$

**lemma** *Prop1B*:

$secure\ c \Longrightarrow \models c : Sec\ (\lambda (r, t). \exists s. (s, c \Downarrow r) \wedge s \approx t) \langle proof \rangle$

**lemma** *Prop1BB*:  $secure\ c \Longrightarrow \exists \Phi. \models c : Sec\ \Phi \langle proof \rangle$

**lemma** *Prop1*:

$(secure\ c) = (\models c : Sec\ (\lambda (r, t). \exists s. (s, c \Downarrow r) \wedge s \approx t)) \langle proof \rangle$

### 3.2 Derivation of the LOW rules

We now derive the interpretation of the LOW rules of Volpano et al's paper according to the constructions given in the paper. (The rules themselves are given later, since they are not yet needed).

**lemma** *CAST*[*rule-format*]:

$G \triangleright c : twiddle \longrightarrow G \triangleright c : Sec\ (\lambda (s, t). s \approx t) \langle proof \rangle$

**lemma** *SKIP*:  $G \triangleright Skip : Sec\ (\lambda (s, t). s \approx t) \langle proof \rangle$

**lemma** *ASSIGN*:

$(\forall s\ ss. s \approx ss \longrightarrow evalE\ e\ s = evalE\ e\ ss) \Longrightarrow$

$G \triangleright (Assign\ x\ e)$

$: (Sec\ (\lambda (s, t). s \approx (update\ t\ x\ (evalE\ e\ t)))) \langle proof \rangle$

**lemma** *COMP*:

$\llbracket G \triangleright c1 : (Sec\ \Phi); G \triangleright c2 : (Sec\ \Psi) \rrbracket \Longrightarrow$

$G \triangleright (Comp\ c1\ c2) : (Sec\ (\lambda (s, t). \exists r. \Phi(r, t) \wedge (\forall w. (r \approx w \longrightarrow \Psi(s, w)))) \langle proof \rangle$

**lemma** *IFF*:

$\llbracket (\forall s\ ss. s \approx ss \longrightarrow evalB\ b\ s = evalB\ b\ ss);$

$G \triangleright c1 : (Sec\ \Phi); G \triangleright c2 : (Sec\ \Psi) \rrbracket \Longrightarrow$

$G \triangleright (Iff\ b\ c1\ c2) : Sec\ (\lambda (s, t). (evalB\ b\ t \longrightarrow \Phi(s, t)) \wedge ((\neg evalB\ b\ t) \longrightarrow \Psi(s, t))) \langle proof \rangle$

We introduce an explicit fixed point construction over the type *TT* of the invariants  $\Phi$ .

**types**  $TT = (State \times State) \Rightarrow bool$

We deliberately introduce a new type here since the agreement with  $VDMAssn$  (modulo currying) is purely coincidental. In particular, in the generalisation for objects in Section 6 the type of invariants will differ from the type of program logic assertions.

**definition**  $FIX::(TT \Rightarrow TT) \Rightarrow TT$

**where**  $FIX \varphi = (\lambda (s,t). \forall \Phi. (\forall ss tt . \varphi \Phi (ss,tt) \longrightarrow \Phi (ss,tt)) \longrightarrow \Phi (s,t))$

**definition**  $Monotone::(TT \Rightarrow TT) \Rightarrow bool$

**where**  $Monotone \varphi = (\forall \Phi \Psi . (\forall s t . \Phi(s,t) \longrightarrow \Psi(s,t)) \longrightarrow (\forall s t . \varphi \Phi (s,t) \longrightarrow \varphi \Psi (s,t)))$

$\langle proof \rangle \langle proof \rangle$

For monotone invariant transformers  $\varphi$ , the construction indeed yields a fixed point.

**lemma** *Fix-lemma*:  $Monotone \varphi \Longrightarrow \varphi (FIX \varphi) = FIX \varphi \langle proof \rangle$

In order to derive the while rule we define the following transformer.

**definition**  $PhiWhileOp::BExpr \Rightarrow TT \Rightarrow TT \Rightarrow TT$

**where**  $PhiWhileOp b \Phi =$

$$(\lambda \Psi . (\lambda(s, t). \\ (evalB b t \longrightarrow (\exists r. \Phi (r, t) \wedge \\ (\forall w. r \approx w \longrightarrow \Psi (s, w)))) \wedge \\ (\neg evalB b t \longrightarrow s \approx t)))$$

Since this operator is monotone, ...

**lemma** *PhiWhileOp-Monotone*:  $Monotone (PhiWhileOp b \Phi) \langle proof \rangle$

we may define its fixed point,

**definition**  $PhiWhile::BExpr \Rightarrow TT \Rightarrow TT$

**where**  $PhiWhile b \Phi = FIX (PhiWhileOp b \Phi)$

which we can use to derive the following rule.

**lemma** *WHILE*:

$$\llbracket (\forall s t. s \approx t \longrightarrow evalB b s = evalB b t); G \triangleright c : (Sec \Phi) \rrbracket \Longrightarrow \\ G \triangleright (While b c) : (Sec (PhiWhile b \Phi)) \langle proof \rangle$$

The operator that given  $\Phi$  returns the invariant occurring in the conclusion of the rule is itself monotone - this is the property required for the rule for procedure invocations.

**lemma** *PhiWhileMonotone*:  $Monotone (\lambda \Phi . PhiWhile b \Phi) \langle proof \rangle$

We now derive an alternative while rule that employs an inductive formulation of a variant that replaces the fixed point construction. This version is given in the paper.

First, the inductive definition of the *var* relation.

**inductive-set**  $var::(BExpr \times TT \times State \times State)$  set

**where**

$varFalse: \llbracket \neg evalB\ b\ t; s \approx t \rrbracket \implies (b, \Phi, s, t):var$   
 $| varTrue: \llbracket evalB\ b\ t; \Phi(r, t); \forall w . r \approx w \longrightarrow (b, \Phi, s, w): var \rrbracket$   
 $\implies (b, \Phi, s, t):var$

It is easy to prove the equivalence of  $var$  and the fixed point:

$\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$   
**lemma**  $FIXvarFIX: (PhiWhile\ b) = (\lambda\ \Phi . (\lambda\ (s, t) . (b, \Phi, s, t):var)) \langle proof \rangle$

From this rule and the rule WHILE above, one may derive the while rule we gave in the paper.

**lemma**  $WHILE-IND:$

$\llbracket (\forall s\ t. s \approx t \longrightarrow evalB\ b\ s = evalB\ b\ t); G \triangleright c : (Sec\ \Phi) \rrbracket \implies$   
 $G \triangleright (While\ b\ c) : (Sec\ (\lambda\ (s, t) . (b, \Phi, s, t):var)) \langle proof \rangle$

Not suprisingly, the construction  $var$  can be shown to be monotone in  $\Phi$ .

$\langle proof \rangle$   
**lemma**  $var-Monotone: Monotone\ (\lambda\ \Phi . (\lambda\ (s, t) . (b, \Phi, s, t):var)) \langle proof \rangle \langle proof \rangle$

The call rule is formulated for an arbitrary fixed point of a monotone transformer.

**lemma**  $CALL:$

$\llbracket (\{Sec(FIX\ \Phi)\} \cup G) \triangleright body : Sec(\Phi\ (FIX\ \Phi)); Monotone\ \Phi \rrbracket \implies$   
 $G \triangleright Call : Sec(FIX\ \Phi) \langle proof \rangle$

### 3.3 Derivation of the HIGH rules

The HIGH rules are easy.

**lemma**  $HIGH-SKIP: G \triangleright Skip : twiddle \langle proof \rangle$

**lemma**  $HIGH-ASSIGN:$

$CONTEXT\ x = high \implies G \triangleright (Assign\ x\ e) : twiddle \langle proof \rangle$

**lemma**  $HIGH-COMP:$

$\llbracket G \triangleright c1 : twiddle; G \triangleright c2 : twiddle \rrbracket$   
 $\implies G \triangleright (Comp\ c1\ c2) : twiddle \langle proof \rangle$

**lemma**  $HIGH-IFF:$

$\llbracket G \triangleright c1 : twiddle; G \triangleright c2 : twiddle \rrbracket$   
 $\implies G \triangleright (Iff\ b\ c1\ c2) : twiddle \langle proof \rangle$

**lemma**  $HIGH-WHILE:$

$\llbracket G \triangleright c : twiddle \rrbracket \implies G \triangleright (While\ b\ c) : twiddle \langle proof \rangle$

**lemma**  $HIGH-CALL:$

$(\{twiddle\} \cup G) \triangleright body : twiddle \implies G \triangleright Call : twiddle \langle proof \rangle$

### 3.4 The type system of Volpano, Smith and Irvine

We now give the type system of Volpano et al. and then prove its embedding into the system of derived rules. First, type systems for expressions and boolean expressions.

**inductive-set**  $VS\text{-}expr :: (Expr \times TP) \text{ set}$   
**where**  
 $VS\text{-}exprVar: CONTEXT\ x = t \implies (varE\ x, t) : VS\text{-}expr$   
 $| VS\text{-}exprVal: (valE\ v, low) : VS\text{-}expr$   
 $| VS\text{-}exprOp: [(e1, t) : VS\text{-}expr; (e2, t) : VS\text{-}expr] \implies (opE\ f\ e1\ e2, t) : VS\text{-}expr$   
 $| VS\text{-}exprHigh: (e, high) : VS\text{-}expr$

**inductive-set**  $VS\text{-}Bexpr :: (BExpr \times TP) \text{ set}$   
**where**  
 $VS\text{-}BexprOp: [(e1, t) : VS\text{-}expr; (e2, t) : VS\text{-}expr] \implies (compB\ f\ e1\ e2, t) : VS\text{-}Bexpr$   
 $| VS\text{-}BexprHigh: (e, high) : VS\text{-}Bexpr$

Next, the core of the type system, the rules for commands.

**inductive-set**  $VS\text{-}com :: (TP \times IMP) \text{ set}$   
**where**

$VS\text{-}comSkip: (pc, Skip) : VS\text{-}com$

$| VS\text{-}comAssHigh:$   
 $CONTEXT\ x = high \implies (pc, Assign\ x\ e) : VS\text{-}com$

$| VS\text{-}comAssLow:$   
 $[(CONTEXT\ x = low; pc = low; (e, low) : VS\text{-}expr)] \implies (pc, Assign\ x\ e) : VS\text{-}com$

$| VS\text{-}comComp:$   
 $[(pc, c1) : VS\text{-}com; (pc, c2) : VS\text{-}com] \implies (pc, Comp\ c1\ c2) : VS\text{-}com$

$| VS\text{-}comIf:$   
 $[(b, pc) : VS\text{-}Bexpr; (pc, c1) : VS\text{-}com; (pc, c2) : VS\text{-}com] \implies (pc, Iff\ b\ c1\ c2) : VS\text{-}com$

$| VS\text{-}comWhile:$   
 $[(b, pc) : VS\text{-}Bexpr; (pc, c) : VS\text{-}com] \implies (pc, While\ b\ c) : VS\text{-}com$

$| VS\text{-}comSub: (high, c) : VS\text{-}com \implies (low, c) : VS\text{-}com$

We define the interpretation of expression typings...

**primrec**  $SemExpr :: Expr \Rightarrow TP \Rightarrow bool$   
**where**  
 $SemExpr\ e\ low = (\forall\ s\ ss. s \approx ss \longrightarrow evalE\ e\ s = evalE\ e\ ss) \mid$   
 $SemExpr\ e\ high = True$

... and show the soundness of the typing rules.

**lemma**  $ExprSound: (e, tp) : VS\text{-}expr \implies SemExpr\ e\ tp \langle proof \rangle$

Likewise for the boolean expressions.

**primrec**  $SemBExpr::BExpr \Rightarrow TP \Rightarrow bool$

**where**

$SemBExpr\ b\ low = (\forall\ s\ ss.\ s \approx ss \longrightarrow evalB\ b\ s = evalB\ b\ ss) \mid$   
 $SemBExpr\ b\ high = True$

**lemma**  $BExprSound: (e, tp): VS-Bexpr \Longrightarrow SemBExpr\ e\ tp \langle proof \rangle$

The proof of the main theorem (called Theorem 2 in our paper) proceeds by induction on  $(t, c) : VS\_com$ .

**theorem**  $VS-com-VDM[rule-format]:$

$(t, c): VS-com \Longrightarrow (t=high \longrightarrow G \triangleright c : twiddle) \wedge$   
 $(t=low \longrightarrow (\exists\ A . G \triangleright c : Sec\ A)) \langle proof \rangle$

The semantic of typing judgements for commands is now the expected one: HIGH commands require initial and final state be low equivalent (i.e. the low variables in the final state can't depend on the high variables of the initial state), while LOW commands must respect the above mentioned security property.

**primrec**  $SemCom::TP \Rightarrow IMP \Rightarrow bool$

**where**

$SemCom\ low\ c = (\forall\ s\ ss\ t\ tt.\ s \approx ss \longrightarrow (s, c \Downarrow t) \longrightarrow$   
 $(ss, c \Downarrow tt) \longrightarrow t \approx tt) \mid$   
 $SemCom\ high\ c = (\forall\ s\ t . (s, c \Downarrow t) \longrightarrow s \approx t)$

Combining theorem  $VS-com-VDM$  with the soundness result of the program logic and the definition of validity yields the soundness of Volpano et al.'s type system.

**theorem**  $VS-SOUND: (t, c): VS-com \Longrightarrow SemCom\ t\ c \langle proof \rangle$

As a further minor result, we prove that all judgements interpreting the low rules indeed yield assertions  $A$  of the form  $A = Sec(\Phi(FIX\ \Phi))$  for some monotone  $\Phi$ .

**inductive-set**  $Deriv :: (VDMAssn\ set \times IMP \times VDMAssn)\ set$

**where**

$D-CAST:$

$(G, c, twiddle): Deriv \Longrightarrow (G, c, Sec\ (\lambda\ (s, t) . s \approx t)) : Deriv$

$\mid D-SKIP: (G, Skip, Sec\ (\lambda\ (s, t) . s \approx t)) : Deriv$

$\mid D-ASSIGN:$

$(\forall\ s\ ss.\ s \approx ss \longrightarrow evalE\ e\ s = evalE\ e\ ss) \Longrightarrow$   
 $(G, Assign\ x\ e, Sec\ (\lambda\ (s, t) . s \approx (update\ t\ x\ (evalE\ e\ t)))) : Deriv$

$\mid D-COMP:$

$\llbracket (G, c1, Sec\ \Phi): Deriv; (G, c2, Sec\ \Psi): Deriv \rrbracket \Longrightarrow$   
 $(G, Comp\ c1\ c2, Sec\ (\lambda\ (s, t) . \exists\ r . \Phi(r, t) \wedge$   
 $(\forall\ w . (r \approx w \longrightarrow \Psi(s, w)))) : Deriv$

| *C-IFF*:  

$$\begin{aligned} & \llbracket (\forall s \ ss. \ s \approx \ ss \longrightarrow \text{eval}B \ b \ s = \text{eval}B \ b \ ss); \\ & \quad (G, \ c1, \ \text{Sec } \Phi):\text{Deriv}; (G, \ c2, \ \text{Sec } \Psi):\text{Deriv} \rrbracket \Longrightarrow \\ & \quad (G, \ \text{Iff } b \ c1 \ c2, \ \text{Sec } (\lambda (s, t) . (\text{eval}B \ b \ t \longrightarrow \Phi(s, t)) \wedge \\ & \quad \quad \quad (\neg \text{eval}B \ b \ t) \longrightarrow \Psi(s, t))):\text{Deriv} \end{aligned}$$

| *D-WHILE*:  

$$\begin{aligned} & \llbracket (\forall s \ ss. \ s \approx \ ss \longrightarrow \text{eval}B \ b \ s = \text{eval}B \ b \ ss); \\ & \quad (G, \ c, \ \text{Sec } \Phi):\text{Deriv} \rrbracket \Longrightarrow \\ & \quad (G, \ \text{While } b \ c, \ \text{Sec } (\text{PhiWhile } b \ \Phi)):\text{Deriv} \end{aligned}$$

| *D-CALL*:  

$$\begin{aligned} & \llbracket (\{\text{Sec}(\text{FIX } \Phi)\} \cup G, \ \text{body}, \ \text{Sec}(\Phi(\text{FIX } \Phi))):\text{Deriv}; \\ & \quad \text{Monotone } \Phi \rrbracket \Longrightarrow \\ & \quad (G, \ \text{Call}, \ \text{Sec}(\text{FIX } \Phi)):\text{Deriv} \end{aligned}$$

| *D-HighSKIP*: $(G, \ \text{Skip}, \ \text{twiddle}): \text{Deriv}$

| *D-HighASSIGN*:  
 $\text{CONTEXT } x = \text{high} \Longrightarrow (G, \ \text{Assign } x \ e, \ \text{twiddle}): \text{Deriv}$

| *D-HighCOMP*:  

$$\begin{aligned} & \llbracket (G, \ c1, \ \text{twiddle}): \text{Deriv}; (G, \ c2, \ \text{twiddle}): \text{Deriv} \rrbracket \Longrightarrow \\ & \quad (G, \ \text{Comp } c1 \ c2, \ \text{twiddle}): \text{Deriv} \end{aligned}$$

| *D-HighIFF*:  

$$\begin{aligned} & \llbracket (G, \ c1, \ \text{twiddle}): \text{Deriv}; (G, \ c2, \ \text{twiddle}): \text{Deriv} \rrbracket \Longrightarrow \\ & \quad (G, \ \text{Iff } b \ c1 \ c2, \ \text{twiddle}): \text{Deriv} \end{aligned}$$

| *D-HighWHILE*:  
 $(G, \ c, \ \text{twiddle}): \text{Deriv} \Longrightarrow (G, \ \text{While } b \ c, \ \text{twiddle}): \text{Deriv}$

| *D-HighCALL*:  
 $(\{\text{twiddle}\} \cup G, \ \text{body}, \ \text{twiddle}): \text{Deriv} \Longrightarrow (G, \ \text{Call}, \ \text{twiddle}): \text{Deriv}$   
 $\langle \text{proof} \rangle$

**lemma** *DerivMono*:  
 $(X, \ c, \ A): \text{Deriv} \Longrightarrow \exists \Phi . A = \text{Sec } (\Phi (\text{FIX } \Phi)) \wedge \text{Monotone } \Phi \langle \text{proof} \rangle$

Also, all rules in the *Deriv* relation are indeed derivable in the program logic.

**lemma** *Deriv-derivable*:  $(G, \ c, \ A): \text{Deriv} \Longrightarrow G \triangleright c: A \langle \text{proof} \rangle$

End of theory VS

**end**

**theory** *ContextVS* **imports** *VS* **begin**

### 3.5 Contextual closure

We show that the notion of security is closed w.r.t. low attacking contexts, i.e. contextual programs into which a secure program can be substituted and which itself employs only *obviously* low variables.

Contexts are **IMP** programs with (multiple) designated holes (represented by constructor *Ctxt\_Here*).

```
datatype CtxtProg =
  Ctxt-Hole
| Ctxt-Skip
| Ctxt-Assign Var Expr
| Ctxt-Comp CtxtProg CtxtProg
| Ctxt-If BExpr CtxtProg CtxtProg
| Ctxt-While BExpr CtxtProg
| Ctxt-Call
```

We let  $C, D$  range over contextual programs. The substitution operation is defined by structural recursion.

```
primrec Fill::CtxtProg  $\Rightarrow$  IMP  $\Rightarrow$  IMP
where
  Fill Ctxt-Hole c = c |
  Fill Ctxt-Skip c = Skip |
  Fill (Ctxt-Assign x e) c = Assign x e |
  Fill (Ctxt-Comp C1 C2) c = Comp (Fill C1 c) (Fill C2 c) |
  Fill (Ctxt-If b C1 C2) c = Iff b (Fill C1 c) (Fill C2 c) |
  Fill (Ctxt-While b C) c = While b (Fill C c) |
  Fill Ctxt-Call c = Call
```

Equally obvious are the definitions of the (syntactically) mentioned variables of arithmetic and boolean expressions.

```
primrec EVars::Expr  $\Rightarrow$  Var set
where
  EVars (varE x) = {x} |
  EVars (valE v) = {} |
  EVars (opE f e1 e2) = EVars e1  $\cup$  EVars e2
```

```
lemma low-Eval[rule-format]:
  ( $\forall x . x \in \text{EVars } e \longrightarrow \text{CONTEXT } x = \text{low}$ )  $\longrightarrow$ 
  ( $\forall s t . s \approx t \longrightarrow \text{evalE } e s = \text{evalE } e t$ )(proof)
```

```
primrec BVars::BExpr  $\Rightarrow$  Var set
where
  BVars (compB f e1 e2) = EVars e1  $\cup$  EVars e2
```

```
lemma low-EvalB[rule-format]:
  ( $\forall x . x \in \text{BVars } b \longrightarrow \text{CONTEXT } x = \text{low}$ )  $\longrightarrow$ 
  ( $\forall s t . s \approx t \longrightarrow \text{evalB } b s = \text{evalB } b t$ )(proof)
```

The variables possibly read from during the evaluation of  $c$  are denoted

by  $Vars\ c$ . Note that in the clause for assignments the variable that is assigned to is not included in the set.

**primrec**  $Vars::IMP \Rightarrow Var\ set$   
**where**  
 $Vars\ Skip = \{\}$  |  
 $Vars\ (Assign\ x\ e) = EVars\ e$  |  
 $Vars\ (Comp\ c\ d) = Vars\ c \cup Vars\ d$  |  
 $Vars\ (While\ b\ c) = BVars\ b \cup Vars\ c$  |  
 $Vars\ (Iff\ b\ c\ d) = BVars\ b \cup Vars\ c \cup Vars\ d$  |  
 $Vars\ Call = \{\}$

For contexts, we define when a set  $X$  of variables is an upper bound for the variables read from.

**primrec**  $CtxtVars::Var\ set \Rightarrow CtxtProg \Rightarrow bool$   
**where**  
 $CtxtVars\ X\ Ctxt-Hole = True$  |  
 $CtxtVars\ X\ Ctxt-Skip = True$  |  
 $CtxtVars\ X\ (Ctxt-Assign\ x\ e) = (EVars\ e \subseteq X)$  |  
 $CtxtVars\ X\ (Ctxt-Comp\ C1\ C2) = (CtxtVars\ X\ C1 \wedge CtxtVars\ X\ C2)$  |  
 $CtxtVars\ X\ (Ctxt-If\ b\ C1\ C2) = (BVars\ b \subseteq X \wedge CtxtVars\ X\ C1$   
 $\quad \wedge\ CtxtVars\ X\ C2)$  |  
 $CtxtVars\ X\ (Ctxt-While\ b\ C) = (BVars\ b \subseteq X \wedge CtxtVars\ X\ C)$  |  
 $CtxtVars\ X\ Ctxt-Call = True$   
 $\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

A constant representing the procedure body with holes.

**consts**  $Ctxt-Body::CtxtProg$

The following predicate expresses that all variables read from by a command  $c$  are contained in the set  $X$  of low variables.

**definition**  $LOW::Var\ set \Rightarrow CtxtProg \Rightarrow bool$   
**where**  $LOW\ X\ C = (CtxtVars\ X\ C \wedge (\forall\ x.\ x : X \longrightarrow CONTEXT\ x = low))$   
 $\langle proof \rangle$

By induction on the maximal height of the operational judgement (hidden in the definition of *secure*) we can prove that the security of  $c$  implies that of  $Fill\ C\ c$ , provided that the context and the procedure-context satisfy the  $LOW$  predicate for some  $X$ , and that the "real" body is obtained by substituting  $c$  into the procedure context.

**lemma**  $secureI-secureFill$ :  
 $\llbracket secure\ c; LOW\ X\ C; LOW\ X\ Ctxt-Body; body = Fill\ Ctxt-Body\ c \rrbracket$   
 $\implies secure\ (Fill\ C\ c) \langle proof \rangle$

Consequently, a (low) variable representing the result of the attacking context does not leak any unwanted information.

**consts**  $res::Var$

**theorem**

```

[[ secure c; LOW X C; LOW X Ctxt-Body; s ≈ ss; s,(Fill C c)↓t;
  ss,(Fill C c)↓tt; body = Fill Ctxt-Body c; CONTEXT res = low]]
⇒ t res = tt res⟨proof⟩

```

End of theory ContextVS

**end**

**theory Lattice imports Main begin**

## 4 Lattices

In preparation of the encoding of the type system of Hunt and Sands, we define some abstract type of lattices, together with the operations  $\perp$ ,  $\sqsubseteq$  and  $\sqcup$ , and some obvious axioms.

**typedecl L**

**axiomatization**

*bottom* :: L **and**

*LEQ* :: L ⇒ L ⇒ bool **and**

*LUB* :: L ⇒ L ⇒ L

**where**

*LAT1*: *LEQ bottom p* **and**

*LAT2*: *LEQ p1 p2 ⇒ LEQ p2 p3 ⇒ LEQ p1 p3* **and**

*LAT3*: *LEQ p (LUB p q)* **and**

*LAT4*: *LUB p q = LUB q p* **and**

*LAT5*: *LUB p (LUB q r) = LUB (LUB p q) r* **and**

*LAT6*: *LEQ x x* **and**

*LAT7*: *p = LUB p p*

End of theory Lattice

**end**

**theory HuntSands imports VDM Lattice begin**

## 5 Flow-sensitivity a la Hunt and Sands

<sup>1</sup> The paper [4] by Hunt and Sands presents a generalisation of the type system of Volpano et al. to flow-sensitivity. Thus, programs such as  $l := h; l := 5$  are not rejected any longer by the type system. Following the description in Section 4 of our paper [2], we embed Hunt and Sands' type system into the program logic given in Section 2.

---

<sup>1</sup>As the Isabelle theory representing this section is dependent only on VDM.thy and Lattice.thy, name conflicts with notions defined in Section 3 are avoided.

## 5.1 General $A; R \Rightarrow S$ -security

Again, we define the type  $TT$  of intermediate formulae  $\Phi$ , and an assertion operator  $Sec$ . The latter is now parametrised not only by the intermediate formulae but also by the (possibly differing) pre- and post-relations  $R$  and  $S$  (both instantiated to  $\approx$  in Section 3), and by a specification  $A$  that directly links pre- and post-states.

**types**  $TT = (State \times State) \Rightarrow bool$

**definition**  $RSsecure :: (State \Rightarrow State \Rightarrow bool) \Rightarrow (State \Rightarrow State \Rightarrow bool) \Rightarrow IMP \Rightarrow bool$   
**where**  $RSsecure R S c = (\forall s t ss tt . R s t \longrightarrow (s, c \Downarrow ss) \longrightarrow (t, c \Downarrow tt) \longrightarrow S ss tt)$

**definition**  $ARSsecure :: VDMAssn \Rightarrow (State \Rightarrow State \Rightarrow bool) \Rightarrow (State \Rightarrow State \Rightarrow bool) \Rightarrow IMP \Rightarrow bool$   
**where**  $ARSsecure A R S c = ((\models c : A) \wedge RSsecure R S c)$

Definition 3 of our paper follows.

**definition**  $Sec :: VDMAssn \Rightarrow (State \Rightarrow State \Rightarrow bool) \Rightarrow (State \Rightarrow State \Rightarrow bool) \Rightarrow TT \Rightarrow VDMAssn$   
**where**  $Sec A R S \Phi s t = (A s t \wedge (\forall r . R s r \longrightarrow \Phi(t, r)) \wedge (\forall r . \Phi(r, s) \longrightarrow S r t))$

With these definitions, we can prove Proposition 4 of our paper.

**lemma**  $Prop4A: \models c : Sec A R S \Phi \Longrightarrow ARSsecure A R S c \langle proof \rangle$

**lemma**  $Prop4B: ARSsecure A R S c \Longrightarrow \models c : Sec A R S (\lambda (r, t) . \exists s . (s, c \Downarrow r) \wedge R s t) \langle proof \rangle$

## 5.2 Basic definitions

Contexts map program variables to lattice elements.

**types**  $CONTEXT = Var \Rightarrow L$

**definition**  $upd :: CONTEXT \Rightarrow Var \Rightarrow L \Rightarrow CONTEXT$   
**where**  $upd G x p = (\lambda y . \text{if } x=y \text{ then } p \text{ else } G y)$

We also define the predicate  $EQ$  which expresses when two states agree on all variables whose entry in a given context is below a certain security level.

**definition**  $EQ :: CONTEXT \Rightarrow L \Rightarrow State \Rightarrow State \Rightarrow bool$   
**where**  $EQ G p = (\lambda s t . \forall x . LEQ (G x) p \longrightarrow s x = t x)$

**lemma**  $EQ\text{-}LEQ: [EQ G p s t; LEQ pp p] \Longrightarrow EQ G pp s t \langle proof \rangle$

The assertion called  $\mathcal{Q}$  in our paper:

**definition**  $Q :: L \Rightarrow CONTEXT \Rightarrow VDMAssn$   
**where**  $Q p H = (\lambda s t . \forall x . (\neg LEQ p (H x)) \longrightarrow t x = s x)$

$Q$  expresses the preservation of values in a single execution, and corresponds to the first clause of Definition 3.2 in [4]. In accordance with this, the following definition of security instantiates the  $A$  position of  $A; R \Rightarrow S$ -security with  $Q$ , while the context-dependent binary state relations are plugged in as the  $R$  and  $S$  components.

**definition**  $secure :: L \Rightarrow CONTEXT \Rightarrow IMP \Rightarrow CONTEXT \Rightarrow bool$   
**where**  $secure\ p\ G\ c\ H = (\forall\ q . ARSsecure\ (Q\ p\ H)\ (EQ\ G\ q)\ (EQ\ H\ q)\ c)$

Indeed, one may show that this notion of security amounts to the conjunction of a unary (i.e. one-execution-)property and a binary (i.e. two-execution-) property, as expressed in Hunt & Sands' Definition 3.2.

**definition**  $secure1 :: L \Rightarrow CONTEXT \Rightarrow IMP \Rightarrow CONTEXT \Rightarrow bool$   
**where**  $secure1\ p\ G\ c\ H = (\forall\ s\ t . (s, c \Downarrow t) \longrightarrow Q\ p\ H\ s\ t)$

**definition**  $secure2 :: L \Rightarrow CONTEXT \Rightarrow IMP \Rightarrow CONTEXT \Rightarrow bool$   
**where**  $secure2\ p\ G\ c\ H = ((\forall\ s\ t\ ss\ tt . (s, c \Downarrow t) \longrightarrow (ss, c \Downarrow tt) \longrightarrow EQ\ G\ p\ s\ ss \longrightarrow EQ\ H\ p\ t\ tt))$

**lemma**  $secureEQUIV$ :

$secure\ p\ G\ c\ H = (\forall\ q . secure1\ p\ G\ c\ H \wedge secure2\ q\ G\ c\ H) \langle proof \rangle$

### 5.3 Type system

The type system of Hunt and Sands – our language formalisation uses a concrete datatype of expressions, so we add the obvious typing rules for expressions and prove the expected evaluation lemmas.

**inductive-set**  $HS-E :: (CONTEXT \times Expr \times L)\ set$

**where**

$HS-E-var: (G, varE\ x, G\ x) : HS-E$   
 $| HS-E-val: (G, valE\ c, bottom) : HS-E$   
 $| HS-E-op: [(G, e1, p1):HS-E; (G, e2, p2):HS-E; p = LUB\ p1\ p2] \Longrightarrow (G, opE\ f\ e1\ e2, p) : HS-E$   
 $| HS-E-sup: [(G, e, p):HS-E; LEQ\ p\ q] \Longrightarrow (G, e, q):HS-E$

**lemma**  $HS-E-eval[rule-format]$ :

$(G, e, t) \in HS-E \Longrightarrow$   
 $\forall\ r\ s\ q. EQ\ G\ q\ r\ s \longrightarrow LEQ\ t\ q \longrightarrow evalE\ e\ r = evalE\ e\ s \langle proof \rangle$

Likewise for boolean expressions:

**inductive-set**  $HS-B :: (CONTEXT \times BExpr \times L)\ set$

**where**

$HS-B-compB: [(G, e1, p1):HS-E; (G, e2, p2):HS-E; p = LUB\ p1\ p2] \Longrightarrow (G, compB\ f\ e1\ e2, p) : HS-B$   
 $| HS-B-sup: [(G, b, p):HS-B; LEQ\ p\ q] \Longrightarrow (G, b, q):HS-B$

**lemma**  $HS-B-eval[rule-format]$ :

$(G, b, t) \in HS-B \Longrightarrow$

$\forall r s pp . EQ G pp r s \longrightarrow LEQ t pp \longrightarrow evalB b r = evalB b s \langle proof \rangle$

The typing rules for commands follow.

**inductive-set**  $HS::(L \times CONTEXT \times IMP \times CONTEXT)$  set

**where**

$HS-Skip: (p, G, Skip, G):HS$

|  $HS-Assign:$

$(G, e, t):HS-E \Longrightarrow (p, G, Assign\ x\ e, upd\ G\ x\ (LUB\ p\ t)):HS$

|  $HS-Seq:$

$\llbracket (p, G, c, K):HS; (p, K, d, H):HS \rrbracket \Longrightarrow (p, G, Comp\ c\ d, H):HS$

|  $HS-If:$

$\llbracket (G, b, t):HS-B; (LUB\ p\ t, G, c, H):HS; (LUB\ p\ t, G, d, H):HS \rrbracket \Longrightarrow (p, G, Iff\ b\ c\ d, H):HS$

|  $HS-If-alg:$

$\llbracket (G, b, p):HS-B; (p, G, c, H):HS; (p, G, d, H):HS \rrbracket \Longrightarrow (p, G, Iff\ b\ c\ d, H):HS$

|  $HS-While:$

$\llbracket (G, b, t):HS-B; (LUB\ p\ t, G, c, H):HS; H=G \rrbracket \Longrightarrow (p, G, While\ b\ c, H):HS$

|  $HS-Sub:$

$\llbracket (pp, GG, c, HH):HS; LEQ\ p\ pp; \forall x . LEQ\ (G\ x)\ (GG\ x); \forall x . LEQ\ (HH\ x)\ (H\ x) \rrbracket \Longrightarrow (p, G, c, H):HS$

Using  $HS-Sub$ , rules  $If$  and  $If-alg$  are inter-derivable.

**lemma**  $IF$ -derivable-from- $If$ -alg:

$\llbracket (G, b, t):HS-B; (LUB\ p\ t, G, c1, H):HS; (LUB\ p\ t, G, c2, H):HS \rrbracket \Longrightarrow (p, G, Iff\ b\ c1\ c2, H):HS$

$\langle proof \rangle$

**lemma**  $If$ -alg-derivable-from- $If$ :

$\llbracket (G, b, p):HS-B; (p, G, c1, H):HS; (p, G, c2, H):HS \rrbracket \Longrightarrow (p, G, Iff\ b\ c1\ c2, H):HS$

$\langle proof \rangle$

An easy induction on typing derivations shows the following property.

**lemma**  $HS-Aux1:$

$(p, G, c, H):HS \Longrightarrow \forall x . LEQ\ (G\ x)\ (H\ x) \vee LEQ\ p\ (H\ x) \langle proof \rangle$

## 5.4 Derived proof rules

In order to show the derivability of the properties given in Theorem 3.3 of Hunt and Sands' paper, we give the following derived proof rules. By

including the  $Q$  property in the  $A$  position of  $Sec$ , we prove both parts of theorem in one proof, and can exploit the first property ( $Q$ ) in the proof of the second.

**lemma SKIP:**

$$X \triangleright Skip : Sec (Q p H) (EQ G q) (EQ G q) \\ (\lambda (s,t) . EQ G q s t)\langle proof \rangle$$

**lemma ASSIGN:**

$$\llbracket H = upd G x (LUB p t); \\ \forall s ss . EQ G t s ss \longrightarrow evalE e s = evalE e ss \rrbracket \\ \implies X \triangleright Assign x e : Sec (Q p H) (EQ G q) (EQ H q) \\ (\lambda (s,t) . \exists r . s = update r x (evalE e r) \wedge EQ G q r t)\langle proof \rangle$$

**lemma COMP:**

$$\llbracket X \triangleright c1 : Sec (Q p K) (EQ G q) (EQ K q) \Phi; \\ X \triangleright c2 : Sec (Q p H) (EQ K q) (EQ H q) \Psi; \\ \forall x . LEQ (G x) (K x) \vee LEQ p (K x); \\ \forall x . LEQ (K x) (H x) \vee LEQ p (H x) \rrbracket \\ \implies X \triangleright Comp c1 c2 : Sec (Q p H) (EQ G q) (EQ H q) \\ (\lambda (x, y) . \exists z . \Phi (z, y) \wedge \\ (\forall w . EQ K q z w \longrightarrow \Psi (x, w)))\langle proof \rangle$$

We distinguish, for any given  $q$ , *parallel* conditionals from *diagonal* ones. Speaking operationally (i.e. in terms of two executions), conditionals of the former kind evaluate the branch condition identically in both executions. The following rule expresses this condition explicitly, in the first side condition. The formula inside the  $Sec$ -operator of the conclusion resembles the conclusion of the VDM rule for conditionals in that the formula chosen depends on the outcome of the branch.

**lemma IF-PARALLEL:**

$$\llbracket \forall s ss . EQ G p s ss \longrightarrow evalB b s = evalB b ss; \\ \forall x . LEQ (G x) (H x) \vee LEQ p (H x); \\ \exists x . LEQ p (H x) \wedge LEQ (H x) q; \\ X \triangleright c1 : Sec (Q p H) (EQ G q) (EQ H q) \Phi; \\ X \triangleright c2 : Sec (Q p H) (EQ G q) (EQ H q) \Psi \rrbracket \\ \implies X \triangleright Iff b c1 c2 : Sec (Q p H) (EQ G q) (EQ H q) \\ (\lambda (r, u) . (evalB b u \longrightarrow \Phi (r, u)) \wedge \\ (\neg evalB b u \longrightarrow \Psi (r, u)))\langle proof \rangle$$

An alternative formulation replaces the first side condition with a typing hypothesis on the branch condition, thus exploiting lemma HS\_B\_eval.

**lemma IF-PARALLEL-tp:**

$$\llbracket (G, b, p) \in HS-B; (p, G, c1, H) \in HS; (p, G, c2, H) \in HS; \\ \exists x . LEQ p (H x) \wedge LEQ (H x) q; \\ X \triangleright c1 : Sec (Q p H) (EQ G q) (EQ H q) \Phi; \\ X \triangleright c2 : Sec (Q p H) (EQ G q) (EQ H q) \Psi \rrbracket \\ \implies X \triangleright Iff b c1 c2 : Sec (Q p H) (EQ G q) (EQ H q) \\ (\lambda (r, u) . (evalB b u \longrightarrow \Phi (r, u)) \wedge \\ (\neg evalB b u \longrightarrow \Psi (r, u)))\langle proof \rangle$$

Diagonal conditionals, in contrast, capture cases where (from the perspective of an observer at level  $q$ ) the two executions may evaluate the branch condition differently. In this case, the formula inside the *Sec*-operator in the conclusion cannot depend upon the branch outcome, so the least common denominator of the two branches must be taken, which is given by the equality condition w.r.t. the post-context  $H$ . A side condition (the first one given in the rule) ensures that indeed no information leaks during the execution of either branch, by relating  $G$  and  $H$ .

**lemma IF-DIAGONAL:**

$$\begin{aligned} & \llbracket \forall x. LEQ (G x) (H x) \vee LEQ p (H x); \\ & \quad \neg (\exists x. LEQ p (H x) \wedge LEQ (H x) q); \\ & \quad X \triangleright c1 : Sec (Q p H) (EQ G q) (EQ H q) \Phi; \\ & \quad X \triangleright c2 : Sec (Q p H) (EQ G q) (EQ H q) \Psi \rrbracket \\ \implies & X \triangleright Iff b c1 c2 : Sec (Q p H) (EQ G q) (EQ H q) \\ & \quad (\lambda (s,t). EQ H q s t)\langle proof \rangle \end{aligned}$$

Again, the first side condition of the rule may be replaced by a typing condition, but now this condition is on the commands (instead of the branch condition) – in fact, a derivation for either branch suffices.

**lemma IF-DIAGONAL-tp:**

$$\begin{aligned} & \llbracket (p, G, c1, H) \in HS \vee (p, G, c2, H) \in HS; \\ & \quad \neg (\exists x. LEQ p (H x) \wedge LEQ (H x) q); \\ & \quad X \triangleright c1 : Sec (Q p H) (EQ G q) (EQ H q) \Phi; \\ & \quad X \triangleright c2 : Sec (Q p H) (EQ G q) (EQ H q) \Psi \rrbracket \\ \implies & X \triangleright Iff b c1 c2 : Sec (Q p H) (EQ G q) (EQ H q) \\ & \quad (\lambda (s,t). EQ H q s t)\langle proof \rangle \end{aligned}$$

Obviously, given  $q$ , any conditional is either parallel or diagonal as the second side conditions of the diagonal rules and the parallel rules are exclusive.

**lemma if-algorithmic:**

$$\begin{aligned} & \llbracket \exists x. LEQ p (H x) \wedge LEQ (H x) q; \\ & \quad \neg (\exists x. LEQ p (H x) \wedge LEQ (H x) q) \rrbracket \\ \implies & False \langle proof \rangle \end{aligned}$$

As in Section 3 we define a fixed point construction, useful for the (parallel) while rule.

**definition FIX::** $(TT \Rightarrow TT) \Rightarrow TT$

**where**  $FIX \varphi = (\lambda (s,t). \forall \Phi . (\forall ss tt . \varphi \Phi (ss, tt) \longrightarrow \Phi (ss, tt)) \longrightarrow \Phi (s, t))$

For monotone invariant transformers, the construction indeed yields a fixed point.

**definition Monotone::** $(TT \Rightarrow TT) \Rightarrow bool$

**where**  $Monotone \varphi = (\forall \Phi \Psi . (\forall s t . \Phi(s,t) \longrightarrow \Psi(s,t)) \longrightarrow (\forall s t . \varphi \Phi (s,t) \longrightarrow \varphi \Psi (s,t)))$

$\langle proof \rangle \langle proof \rangle$  **lemma Fix-lemma:**  $Monotone \varphi \implies \varphi (FIX \varphi) = FIX \varphi \langle proof \rangle$

Next, the definition of a while-operator.

**definition** *PhiWhilePOp*::

$$VDMAssn \Rightarrow BExpr \Rightarrow TT \Rightarrow TT \Rightarrow TT$$

**where** *PhiWhilePOp*  $A b \Phi =$

$$(\lambda \Psi . (\lambda(r, u). (evalB b u \longrightarrow (\exists z. \Phi(z, u) \wedge (\forall w. A z w \longrightarrow \Psi(r, w)))) \wedge ((\neg evalB b u) \longrightarrow A r u)))$$

This operator is monotone in  $\Phi$ .

**lemma** *PhiWhilePOp-Monotone*: *Monotone* (*PhiWhilePOp*  $A b \Phi$ )*<proof>*

Therefore, we can define the following fixed point.

**definition** *PhiWhileP*:: *VDMAssn*  $\Rightarrow$  *BExpr*  $\Rightarrow$  *TT*  $\Rightarrow$  *TT*

**where** *PhiWhileP*  $A b \Phi = FIX$  (*PhiWhilePOp*  $A b \Phi$ )

As as a function on  $\phi$ , this *PhiWhileP* is itself monotone in  $\phi$ :

**lemma** *PhiWhilePMonotone*: *Monotone* ( $\lambda \Phi .$  *PhiWhileP*  $A b \Phi$ )*<proof>*

Now the rule for parallel while loops, i.e. loops where the branch condition evaluates identically in both executions.

**lemma** *WHILE-PARALLEL*:

$$\begin{aligned} & \llbracket X \triangleright c : Sec (Q p G) (EQ G q) (EQ G q) \Phi; \\ & \quad \forall s ss . EQ G p s ss \longrightarrow evalB b s = evalB b ss; LEQ p q \rrbracket \\ \implies & X \triangleright While b c : Sec (Q p G) (EQ G q) (EQ G q) \\ & \quad (PhiWhileP (EQ G q) b \Phi) \langle proof \rangle \end{aligned}$$

The side condition regarding the evaluation of the branch condition may be replaced by a typing hypothesis, thanks to lemma *HS-B-eval*.

**lemma** *WHILE-PARALLEL-tp*:

$$\begin{aligned} & \llbracket X \triangleright c : Sec (Q p G) (EQ G q) (EQ G q) \Phi; \\ & \quad (G, b, p) \in HS-B; LEQ p q \rrbracket \\ \implies & X \triangleright While b c : Sec (Q p G) (EQ G q) (EQ G q) \\ & \quad (PhiWhileP (EQ G q) b \Phi) \langle proof \rangle \end{aligned}$$

One may also give an inductive formulation of *FIX*:

**inductive-set** *var*::(*BExpr*  $\times$  *VDMAssn*  $\times$  *TT*  $\times$  *State*  $\times$  *State*) *set*

**where**

*varFalse*:

$$\llbracket \neg evalB b t; A s t \rrbracket \implies (b, A, \Phi, s, t):var$$

| *varTrue*:

$$\begin{aligned} & \llbracket evalB b t; \Phi(r, t); (\forall w . A r w \longrightarrow \\ & \quad (b, A, \Phi, s, w): var) \rrbracket \implies (b, A, \Phi, s, t):var \\ & \langle proof \rangle \langle proof \rangle \end{aligned}$$

The inductive formulation and the fixed point formulation are equivalent.

*<proof>* *<proof>* **lemma** *FIXvarFIX*:

$$PhiWhileP A b = (\lambda \Phi . (\lambda (s, t) . (b, A, \Phi, s, t):var)) \langle proof \rangle$$

Thus, the above while rule may also be written using the inductive formulation.

**lemma** *WHILE-PARALLEL-IND*:

$$\begin{aligned} & \llbracket X \triangleright c : \text{Sec } (Q \ p \ G) \ (EQ \ G \ q) \ (EQ \ G \ q) \ \Phi; \\ & \quad \forall s \ ss . EQ \ G \ p \ s \ ss \longrightarrow evalB \ b \ s = evalB \ b \ ss; LEQ \ p \ q \rrbracket \Longrightarrow \\ & X \triangleright \text{While } b \ c : (\text{Sec } (Q \ p \ G) \ (EQ \ G \ q) \ (EQ \ G \ q) \\ & \quad (\lambda (s,t) . (b, EQ \ G \ q, \Phi, s, t):var)) \langle proof \rangle \end{aligned}$$

Again, we may replace the side condition regarding the branch condition by a typing hypothesis.

**lemma** *WHILE-PARALLEL-IND-tp*:

$$\begin{aligned} & \llbracket X \triangleright c : \text{Sec } (Q \ p \ G) \ (EQ \ G \ q) \ (EQ \ G \ q) \ \Phi; \\ & \quad (G, b, p) \in HS\text{-}B; LEQ \ p \ q \rrbracket \Longrightarrow \\ & X \triangleright (\text{While } b \ c) : \\ & \quad (\text{Sec } (Q \ p \ G) \ (EQ \ G \ q) \ (EQ \ G \ q) \ (\lambda (s,t) . (b, EQ \ G \ q, \Phi, s, t):var)) \langle proof \rangle \langle proof \rangle \end{aligned}$$

Of course, the inductive formulation is also monotone:

**lemma** *var-MonotoneInPhi*:

$$\text{Monotone } (\lambda \Phi . (\lambda (s,t) . (b, A, \Phi, s, t):var)) \langle proof \rangle \langle proof \rangle$$

In order to derive a diagonal while rule, we directly define an inductive relation that calculates the transitive closure of relation  $A$ , such that all but the last state evaluate  $b$  to *True*.

**inductive-set**  $varD :: (BExpr \times VDMAssn \times State \times State) \text{ set}$

**where**

$$\begin{aligned} & varDFalse: \llbracket \neg evalB \ b \ s; A \ s \ t \rrbracket \Longrightarrow (b, A, s, t):varD \\ & | varDTrue: \llbracket evalB \ b \ s; A \ s \ w; (b, A, w, t):varD \rrbracket \Longrightarrow (b, A, s, t):varD \end{aligned}$$

Here is the obvious definition of transitivity for assertions.

**definition** *transitive* ::  $VDMAssn \Rightarrow bool$

**where** *transitive*  $P = (\forall x \ y \ z . P \ x \ y \longrightarrow P \ y \ z \longrightarrow P \ x \ z)$

The inductive relation satisfies the following property.

**lemma** *varD-transitive*[*rule-format*]:

$$(b, A, s, t):varD \Longrightarrow \text{transitive } A \longrightarrow A \ s \ t \langle proof \rangle$$

On the other hand, the assertion  $Q$  defined above is transitive,

**lemma** *Q-transitive*: *transitive*  $(Q \ q \ G) \langle proof \rangle$

and is hence respected by the inductive closure:

**lemma** *varDQ*:  $(b, Q \ q \ G, s, t):varD \Longrightarrow Q \ q \ G \ s \ t \langle proof \rangle$

The diagonal while rule has a conclusion that is independent of  $\phi$ .

**lemma** *WHILE-DIAGONAL*:

$$\begin{aligned} & \llbracket X \triangleright c : \text{Sec } (Q \ p \ G) \ (EQ \ G \ q) \ (EQ \ G \ q) \ \Phi; \neg LEQ \ p \ q \rrbracket \\ & \quad \Longrightarrow X \triangleright \text{While } b \ c : \text{Sec } (Q \ p \ G) \ (EQ \ G \ q) \ (EQ \ G \ q) \\ & \quad \quad (\lambda (s,t) . EQ \ G \ q \ s \ t) \langle proof \rangle \end{aligned}$$

$varD$  is monotone in the assertion position.

**lemma** *varDMonotoneInAssertion*[*rule-format*]:

$$(b, A, s, t) \in varD \implies (\forall s t. A s t \longrightarrow B s t) \longrightarrow (b, B, s, t) \in varD \langle proof \rangle \langle proof \rangle$$

Finally, the subsumption rule.

**lemma** *SUB*:

$$\begin{aligned} & \llbracket LEQ p pp; \forall x. LEQ (G x) (GG x); \forall x. LEQ (HH x) (H x); \\ & \quad X \triangleright c : Sec (Q pp HH) (EQ GG q) (EQ HH q) \Phi \rrbracket \\ & \implies X \triangleright c : Sec (Q p H) (EQ G q) (EQ H q) \Phi \langle proof \rangle \end{aligned}$$

## 5.5 Soundness results

$\langle proof \rangle$

An induction on the typing rules now proves the main theorem which was called Theorem 4 in [2].

**theorem** *Theorem4*[*rule-format*]:

$$(p, G, c, H) : HS \implies (\exists \Phi . X \triangleright c : (Sec (Q p H) (EQ G q) (EQ H q) \Phi)) \langle proof \rangle$$

By the construction of the operator *Sec* (lemmas *Prop4A* and *Prop4A* in Section 5.1) we obtain the soundness property with respect to the operational semantics, i.e. the result stated as Theorem 3.3 in [4].

**theorem** *HuntSands33*:  $(p, G, c, H) : HS \implies secure\ p\ G\ c\ H \langle proof \rangle$

Both parts of this theorem may also be shown individually. We factor both proofs by the program logic.

**lemma** *Sec1-deriv*:  $(p, G, c, H) : HS \implies X \triangleright c : (Q p H) \langle proof \rangle \langle proof \rangle$

**theorem** *HuntSands33-1*:  $(p, G, c, H) : HS \implies secure1\ p\ G\ c\ H \langle proof \rangle$

**lemma** *Sec2-deriv*:

$$(p, G, c, H) : HS \implies (\exists A . X \triangleright c : (Sec (Q p H) (EQ G q) (EQ H q) A)) \langle proof \rangle \langle proof \rangle$$

**theorem** *HuntSands33-2*:  $(p, G, c, H) : HS \implies secure2\ q\ G\ c\ H \langle proof \rangle$

Again, the call rule is formulated for an arbitrary fixed point of a monotone transformer.

**lemma** *CALL*:

$$\begin{aligned} & \llbracket (\{B\} \cup X) \triangleright body : Sec A R S (\varphi(FIX \varphi)); \\ & \quad Monotone \varphi; B = Sec A R S (FIX \varphi) \rrbracket \\ & \implies X \triangleright Call : B \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \end{aligned}$$

As in Section 3, we define a formal derivation system comprising all derived rules and show that all derivable judgements are of the for  $Sec(\Phi)$  for some monotone  $\Phi$ .

**inductive-set** *Deriv*::  $(VDMAssn\ set \times IMP \times VDMAssn)\ set$

**where**

*D-SKIP*:

$\Omega = (\lambda (s,t). EQ G q s t)$   
 $\implies (X, Skip, Sec (Q p H) (EQ G q) (EQ G q) \Omega) : Deriv$

| *D-ASSIGN*:  
 $\llbracket H = upd G x (LUB p t);$   
 $\forall s ss . EQ G t s ss \longrightarrow evalE e s = evalE e ss;$   
 $\Omega = (\lambda (s, t) . \exists r . s = update r x (evalE e r) \wedge EQ G q r t) \rrbracket$   
 $\implies (X, Assign x e, Sec (Q p H) (EQ G q) (EQ H q) \Omega) : Deriv$

| *D-COMP*:  
 $\llbracket (X, c, Sec (Q p K) (EQ G q) (EQ K q) \Phi) : Deriv;$   
 $(X, d, Sec (Q p H) (EQ K q) (EQ H q) \Psi) : Deriv;$   
 $\forall x . LEQ (G x) (K x) \vee LEQ p (K x);$   
 $\forall x . LEQ (K x) (H x) \vee LEQ p (H x);$   
 $\Omega = (\lambda (x, y) . \exists z . \Phi(z,y) \wedge (\forall w . EQ K q z w \longrightarrow \Psi(x,w))) \rrbracket$   
 $\implies (X, Comp c d, Sec (Q p H) (EQ G q) (EQ H q) \Omega) : Deriv$

| *D-IF-PARALLEL*:  
 $\llbracket \forall s ss . EQ G p s ss \longrightarrow evalB b s = evalB b ss;$   
 $\forall x . LEQ (G x) (H x) \vee LEQ p (H x);$   
 $\exists x . LEQ p (H x) \wedge LEQ (H x) q;$   
 $(X, c, Sec (Q p H) (EQ G q) (EQ H q) \Phi) : Deriv;$   
 $(X, d, Sec (Q p H) (EQ G q) (EQ H q) \Psi) : Deriv;$   
 $\Omega = (\lambda (r, u) . (evalB b u \longrightarrow \Phi(r,u)) \wedge$   
 $(\neg evalB b u \longrightarrow \Psi(r,u))) \rrbracket$   
 $\implies (X, Iff b c d, Sec (Q p H) (EQ G q) (EQ H q) \Omega) : Deriv$

| *D-IF-DIAGONAL*:  
 $\llbracket \forall x . LEQ (G x) (H x) \vee LEQ p (H x);$   
 $\neg (\exists x . LEQ p (H x) \wedge LEQ (H x) q);$   
 $(X, c, Sec (Q p H) (EQ G q) (EQ H q) \Phi) : Deriv;$   
 $(X, d, Sec (Q p H) (EQ G q) (EQ H q) \Psi) : Deriv;$   
 $\Omega = (\lambda (s,t) . EQ H q s t) \rrbracket$   
 $\implies (X, Iff b c d, Sec (Q p H) (EQ G q) (EQ H q) \Omega) : Deriv$

| *D-WHILE-PARALLEL*:  
 $\llbracket (X, c, Sec (Q p G) (EQ G q) (EQ G q) \Phi) : Deriv;$   
 $\forall s ss . EQ G p s ss \longrightarrow evalB b s = evalB b ss; LEQ p q;$   
 $\Omega = (\lambda (s,t) . (b, EQ G q, \Phi, s, t) : var) \rrbracket$   
 $\implies (X, While b c, Sec (Q p G) (EQ G q) (EQ G q) \Omega) : Deriv$

| *D-WHILE-DIAGONAL*:  
 $\llbracket (X, c, Sec (Q p G) (EQ G q) (EQ G q) \Phi) : Deriv; \neg LEQ p q;$   
 $\Omega = (\lambda (s,t) . EQ G q s t) \rrbracket$   
 $\implies (X, While b c, Sec (Q p G) (EQ G q) (EQ G q) \Omega) : Deriv$

| *D-SUB*:  
 $\llbracket LEQ p pp; \forall x . LEQ (G x) (GG x); \forall x . LEQ (HH x) (H x);$   
 $(X, c, Sec (Q pp HH) (EQ GG q) (EQ HH q) \Phi) : Deriv \rrbracket$

$\implies (X, c, \text{Sec } (Q \ p \ H) \ (EQ \ G \ q) \ (EQ \ H \ q) \ \Phi) : \text{Deriv}$

| *D-CALL*:

$(\{A\} \cup X, \text{body}, A) : \text{Deriv} \implies (X, \text{Call}, A) : \text{Deriv}$   
 $\langle \text{proof} \rangle$

**lemma** *DerivMono*:

$(X, c, B) : \text{Deriv} \implies$   
 $\exists A \ R \ S \ \varphi . B = \text{Sec } A \ R \ S \ (\varphi \ (\text{FIX } \varphi)) \wedge \text{Monotone } \varphi \langle \text{proof} \rangle$

Also, the *Deriv* is indeed a subsystem of the program logic.

**theorem** *Deriv-derivable*:  $(X, c, A) : \text{Deriv} \implies X \triangleright c : A \langle \text{proof} \rangle$

End of theory HuntSands

**end**

**theory** *OBJ* imports *Main* begin

## 6 Base-line non-interference with objects

We now extend the encoding for base-line non-interference to a language with objects. The development follows the structure of Sections 1 to 3. Syntax and operational semantics are defined in Section 6.1, the axiomatic semantics in Section 6.2. The generalised definition of non-interference is given in 6.4, the derived proof rules in Section 6.5, and a type system in the style of Volpano et al. in Section 6.6. Finally, Section 6.7 concludes with results on contextual closure.

### 6.1 Syntax and operational semantics

First, some operations for association lists

**primrec** *lookup* ::  $('a \times 'b) \text{ list} \Rightarrow 'a \Rightarrow 'b \text{ option}$

**where**

$\text{lookup } [] \ l = \text{None} \ |$

$\text{lookup } (h \ \# \ t) \ l = (\text{if } (\text{fst } h) = l \ \text{then } \text{Some } (\text{snd } h) \ \text{else } \text{lookup } t \ l)$

**definition** *Dom*::  $('a \times 'b) \text{ list} \Rightarrow 'a \text{ set}$

**where**  $\text{Dom } L = \{l . \exists a . \text{lookup } L \ l = \text{Some } a\}$

$\langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$

Abstract types of variables, class names, field names, and locations.

**typedecl** *Var*

**typedecl** *Class*

**typedecl** *Field*  
**typedecl** *Location*

References are either null or a location. Values are either integers or references.

**datatype** *Ref* = *Nullref* | *Loc Location*

**datatype** *Val* = *RVal Ref* | *IVal int*

The heap is a finite map from locations to objects. Objects have a dynamic class and a field map.

**types** *Object* = *Class* × ((*Field* × *Val*) *list*)

**types** *Heap* = (*Location* × *Object*) *list*

Stores contain values for all variables, and states are pairs of stores and heaps.

**types** *Store* = *Var* ⇒ *Val*

**definition** *update* :: *Store* ⇒ *Var* ⇒ *Val* ⇒ *Store*

**where** *update* *s* *x* *v* = (λ *y* . if *x=y* then *v* else *s y*)

**types** *State* = *Store* × *Heap*

Arithmetic and boolean expressions are as before.

**datatype** *Expr* =  
  *varE Var*  
  | *valE Val*  
  | *opE Val* ⇒ *Val* ⇒ *Val Expr Expr*

**datatype** *BExpr* = *compB Val* ⇒ *Val* ⇒ *bool Expr Expr*

The same applies to their semantics.

**primrec** *evalE*::*Expr* ⇒ *Store* ⇒ *Val*

**where**

*evalE* (*varE* *x*) *s* = *s x* |

*evalE* (*valE* *v*) *s* = *v* |

*evalE* (*opE* *f* *e1* *e2*) *s* = *f* (*evalE* *e1* *s*) (*evalE* *e2* *s*)

**primrec** *evalB*::*BExpr* ⇒ *Store* ⇒ *bool*

**where**

*evalB* (*compB* *f* *e1* *e2*) *s* = *f* (*evalE* *e1* *s*) (*evalE* *e2* *s*)

The category of commands is extended by instructions for allocating a fresh object, obtaining a value from a field and assigning a value to a field.

**datatype** *OBJ* =  
  *Skip*  
  | *Assign Var Expr*  
  | *New Var Class*

| *Get Var Var Field*  
 | *Put Var Field Expr*  
 | *Comp OBJ OBJ*  
 | *While BExpr OBJ*  
 | *Iff BExpr OBJ OBJ*  
 | *Call*

The body of the procedure is identified by the same constant as before.

**consts** *body* :: *OBJ*

The operational semantics is again a standard big-step relation.

**inductive-set** *Semn* :: (*State* × *OBJ* × *nat* × *State*) *set*

**where**

*SemSkip*:  $s=t \implies (s, \text{Skip}, 1, t):\text{Semn}$

| *SemAssign*:  
 $\llbracket t = (\text{update } (fst\ s)\ x\ (\text{evalE } e\ (fst\ s))),\ snd\ s \rrbracket$   
 $\implies (s, \text{Assign } x\ e,\ 1,\ t):\text{Semn}$

| *SemNew*:  
 $\llbracket l \notin \text{Dom } (snd\ s);$   
 $t = (\text{update } (fst\ s)\ x\ (\text{RVal } (\text{Loc } l)),\ (l, (C, [])) \# (snd\ s)) \rrbracket$   
 $\implies (s, \text{New } x\ C,\ 1,\ t):\text{Semn}$

| *SemGet*:  
 $\llbracket (fst\ s)\ y = \text{RVal } (\text{Loc } l); \text{lookup } (snd\ s)\ l = \text{Some } (C, \text{Flds});$   
 $\text{lookup } \text{Flds } F = \text{Some } v; t = (\text{update } (fst\ s)\ x\ v,\ snd\ s) \rrbracket$   
 $\implies (s, \text{Get } x\ y\ F,\ 1,\ t):\text{Semn}$

| *SemPut*:  
 $\llbracket (fst\ s)\ x = \text{RVal } (\text{Loc } l); \text{lookup } (snd\ s)\ l = \text{Some } (C, \text{Flds});$   
 $t = (fst\ s,\ (l, (C, (F, \text{evalE } e\ (fst\ s)) \# \text{Flds})) \# (snd\ s)) \rrbracket$   
 $\implies (s, \text{Put } x\ F\ e,\ 1,\ t):\text{Semn}$

| *SemComp*:  
 $\llbracket (s, c, n, r):\text{Semn}; (r, d, m, t):\text{Semn}; k=(\max\ n\ m)+1 \rrbracket$   
 $\implies (s, \text{Comp } c\ d,\ k,\ t):\text{Semn}$

| *SemWhileT*:  
 $\llbracket \text{evalB } b\ (fst\ s); (s, c, n, r):\text{Semn}; (r, \text{While } b\ c,\ m, t):\text{Semn}; k=((\max\ n\ m)+1) \rrbracket$   
 $\implies (s, \text{While } b\ c,\ k,\ t):\text{Semn}$

| *SemWhileF*:  
 $\llbracket \neg (\text{evalB } b\ (fst\ s)); t=s \rrbracket \implies (s, \text{While } b\ c,\ 1,\ t):\text{Semn}$

| *SemTrue*:  
 $\llbracket \text{evalB } b\ (fst\ s); (s, c1, n, t):\text{Semn} \rrbracket$   
 $\implies (s, \text{Iff } b\ c1\ c2,\ n+1,\ t):\text{Semn}$

| *SemFalse*:  
 $\llbracket \neg (evalB\ b\ (fst\ s)); (s, c2, n, t):Semn \rrbracket$   
 $\implies (s, Iff\ b\ c1\ c2, n+1, t):Semn$

| *SemCall*:  $\llbracket (s, body, n, t):Semn \rrbracket \implies (s, Call, n+1, t):Semn$

**abbreviation**

*SemN* :: [State, OBJ, nat, State]  $\Rightarrow$  bool ( - , -  $\rightarrow$  - )

**where**

$s, c \rightarrow_n t \equiv (s, c, n, t) : Semn$

Often, the height index does not matter, so we define a notion hiding it.

**definition**

*Sem* :: [State, OBJ, State]  $\Rightarrow$  bool ( - , -  $\Downarrow$  - 1000)

**where**  $s, c \Downarrow t = (\exists\ n. s, c \rightarrow_n t)$

**inductive-cases** *Sem-eval-cases*:

$s, Skip \rightarrow_n t$   
 $s, (Assign\ x\ e) \rightarrow_n t$   
 $s, (New\ x\ C) \rightarrow_n t$   
 $s, (Get\ x\ y\ F) \rightarrow_n t$   
 $s, (Put\ x\ F\ e) \rightarrow_n t$   
 $s, (Comp\ c1\ c2) \rightarrow_n t$   
 $s, (While\ b\ c) \rightarrow_n t$   
 $s, (Iff\ b\ c1\ c2) \rightarrow_n t$   
 $s, Call \rightarrow_n t$   
 $\langle proof \rangle$  **lemma** *Sem-no-zero-height-derivs*:  $(s, c \rightarrow_0 t) \implies False \langle proof \rangle$

Determinism does not hold as allocation is nondeterministic.

End of theory OBJ

**end**

**theory** *VDM-OBJ* **imports** *OBJ* **begin**

**6.2 Program logic**

Apart from the addition of proof rules for the three new instructions, this section is essentially identical to Section 2.

**6.2.1 Assertions and their semantic validity**

Assertions are binary state predicates, as before.

**types** *Assn* = State  $\Rightarrow$  State  $\Rightarrow$  bool

**definition** *VDM-validn* :: nat  $\Rightarrow$  OBJ  $\Rightarrow$  Assn  $\Rightarrow$  bool

( $\models - : -$  50)  
**where** ( $\models_n c : A$ ) = ( $\forall m . m \leq n \longrightarrow (\forall s t . (s, c \rightarrow_m t) \longrightarrow A s t)$ )

**definition** *VDM-valid* :: *OBJ*  $\Rightarrow$  *Assn*  $\Rightarrow$  *bool*

( $\models - : -$  50)  
**where** ( $\models c : A$ ) = ( $\forall s t . (s, c \Downarrow t) \longrightarrow A s t$ )

**lemma** *VDM-valid-validn*:  $\models c : A \Longrightarrow \models_n c : A \langle \text{proof} \rangle$

**lemma** *VDM-validn-valid*: ( $\forall n . \models_n c : A$ )  $\Longrightarrow \models c : A \langle \text{proof} \rangle$

**lemma** *VDM-lowerm*:  $\llbracket \models_n c : A ; m < n \rrbracket \Longrightarrow \models_m c : A \langle \text{proof} \rangle$

**definition** *Ctxt-validn* :: *nat*  $\Rightarrow$  (*Assn set*)  $\Rightarrow$  *bool*

( $\models - -$  50)  
**where** ( $\models_n G$ ) = ( $\forall m . m \leq n \longrightarrow (\forall A . A \in G \longrightarrow \models_n \text{Call} : A)$ )

**definition** *Ctxt-valid* :: *Assn set*  $\Rightarrow$  *bool* ( $\models -$  50)

**where** ( $\models G$ ) = ( $\forall A . A \in G \longrightarrow \models \text{Call} : A$ )

**lemma** *Ctxt-valid-validn*:  $\models G \Longrightarrow \models_n G \langle \text{proof} \rangle$

**lemma** *Ctxt-validn-valid*: ( $\forall n . \models_n G$ )  $\Longrightarrow \models G \langle \text{proof} \rangle$

**lemma** *Ctxt-lowerm*:  $\llbracket \models_n G ; m < n \rrbracket \Longrightarrow \models_m G \langle \text{proof} \rangle$

**definition** *valid* :: (*Assn set*)  $\Rightarrow$  *OBJ*  $\Rightarrow$  *Assn*  $\Rightarrow$  *bool*

( $- \models - : -$  50)  
**where** ( $G \models c : A$ ) = (*Ctxt-valid*  $G \longrightarrow$  *VDM-valid*  $c A$ )

**definition** *validn* :: (*Assn set*)  $\Rightarrow$  *nat*  $\Rightarrow$  *OBJ*  $\Rightarrow$  *Assn*  $\Rightarrow$  *bool*

( $- \models - - : -$  50)  
**where** ( $G \models_n c : A$ ) = ( $\models_n G \longrightarrow \models_n c : A$ )

**lemma** *validn-valid*: ( $\forall n . G \models_n c : A$ )  $\Longrightarrow G \models c : A \langle \text{proof} \rangle$

**lemma** *ctxt-consn*:  $\llbracket \models_n G ; \models_n \text{Call} : A \rrbracket \Longrightarrow \models_n \{A\} \cup G \langle \text{proof} \rangle$

## 6.2.2 Proof system

**inductive-set** *VDM-proof* :: (*Assn set*  $\times$  *OBJ*  $\times$  *Assn*) *set*

**where**

*VDMSkip*: (*G*, *Skip*,  $\lambda s t . t = s$ ): *VDM-proof*

| *VDMAssign*:

(*G*, *Assign*  $x e$ ,  
 $\lambda s t . t = (\text{update } (fst s) x (\text{evalE } e (fst s)), snd s)$ ): *VDM-proof*

| *VDMNew*:

(*G*, *New*  $x C$ ,  
 $\lambda s t . \exists l . l \notin \text{Dom } (snd s) \wedge$   
 $t = (\text{update } (fst s) x (\text{RVal } (\text{Loc } l)),$   
 $(l, (C, [])) \# (snd s))$ ): *VDM-proof*

| *VDMGet*:

(*G*, *Get*  $x y F$ ,

$$\lambda s t . \exists l C Flds v . (fst s) y = RVal(Loc l) \wedge$$

$$lookup (snd s) l = Some(C, Flds) \wedge$$

$$lookup Flds F = Some v \wedge$$

$$t = (update (fst s) x v, snd s): VDM-proof$$

| *VDMPut*:

$$(G, Put x F e,$$

$$\lambda s t . \exists l C Flds . (fst s) x = RVal(Loc l) \wedge$$

$$lookup (snd s) l = Some(C, Flds) \wedge$$

$$t = (fst s,$$

$$(l, (C, (F, evalE e (fst s)) \# Flds))$$

$$\# (snd s)): VDM-proof$$

| *VDMComp*:

$$\llbracket (G, c, A): VDM-proof; (G, d, B): VDM-proof \rrbracket \implies$$

$$(G, Comp c d, \lambda s t . \exists r . A s r \wedge B r t): VDM-proof$$

| *VDMIff*:

$$\llbracket (G, c, A): VDM-proof; (G, d, B): VDM-proof \rrbracket \implies$$

$$(G, Iff b c d,$$

$$\lambda s t . (((evalB b (fst s)) \longrightarrow A s t) \wedge$$

$$((\neg (evalB b (fst s))) \longrightarrow B s t))): VDM-proof$$

| *VDMWhile*:

$$\llbracket (G, c, B): VDM-proof;$$

$$\forall s . (\neg evalB b (fst s)) \longrightarrow A s s;$$

$$\forall s r t . evalB b (fst s) \longrightarrow B s r \longrightarrow A r t \longrightarrow A s t \rrbracket$$

$$\implies (G, While b c, \lambda s t . A s t \wedge \neg (evalB b (fst t))): VDM-proof$$

| *VDMCall*:

$$(\{A\} \cup G, body, A): VDM-proof \implies (G, Call, A): VDM-proof$$

| *VDMAx*:  $A \in G \implies (G, Call, A): VDM-proof$

| *VDMConseq*:

$$\llbracket (G, c, A): VDM-proof; \forall s t . A s t \longrightarrow B s t \rrbracket \implies$$

$$(G, c, B): VDM-proof$$

**abbreviation** *VDM-deriv* :: [Assn set, OBJ, Assn]  $\Rightarrow$  bool  
 (-  $\triangleright$  - : - [100,100] 50)

**where**  $G \triangleright c : A == (G, c, A) \in VDM-proof$

The while-rule is in fact inter-derivable with the following rule.

**lemma** *Hoare-While*:

$$G \triangleright c : (\lambda s t . \forall r . evalB b (fst s) \longrightarrow I s r \longrightarrow I t r) \implies$$

$$G \triangleright While b c : (\lambda s t . \forall r . I s r \longrightarrow (I t r \wedge \neg evalB b (fst t)))$$

*<proof>*

Here's the proof in the opposite direction.

**lemma** *VDMWhile-derivable*:

$$\begin{aligned} & \llbracket G \triangleright c : B; \forall s . (\neg \text{eval} B b (\text{fst } s)) \longrightarrow A s s; \\ & \quad \forall s r t . \text{eval} B b (\text{fst } s) \longrightarrow B s r \longrightarrow A r t \longrightarrow A s t \rrbracket \\ & \implies G \triangleright (\text{While } b c) : (\lambda s t . A s t \wedge \neg (\text{eval} B b (\text{fst } t))) \\ & \langle \text{proof} \rangle \end{aligned}$$

### 6.2.3 Soundness

$\langle \text{proof} \rangle$

The following auxiliary lemma for loops is proven by induction on  $n$ .

**lemma** *SoundWhile*[*rule-format*]:

$$\begin{aligned} & (\forall n . G \models_n c : B) \\ & \longrightarrow (\forall s . (\neg \text{eval} B b (\text{fst } s)) \longrightarrow A s s) \\ & \longrightarrow (\forall s . \text{eval} B b (\text{fst } s) \\ & \quad \longrightarrow (\forall r . B s r \longrightarrow (\forall t . A r t \longrightarrow A s t))) \\ & \longrightarrow G \models_n (\text{While } b c) : (\lambda s t . A s t \wedge \neg \text{eval} B b (\text{fst } t)) \langle \text{proof} \rangle \end{aligned}$$

**lemma** *SoundCall*[*rule-format*]:

$$\llbracket \forall n . \models_n (\{A\} \cup G) \longrightarrow \models_n \text{body} : A \rrbracket \implies \models_n G \longrightarrow \models_n \text{Call} : A \langle \text{proof} \rangle$$

**lemma** *VDM-Sound-n*:  $G \triangleright c : A \implies (\forall n . G \models_n c : A) \langle \text{proof} \rangle$

**theorem** *VDM-Sound*:  $G \triangleright c : A \implies G \models c : A \langle \text{proof} \rangle$

A simple corollary is soundness w.r.t. an empty context.

**lemma** *VDM-Sound-emptyCtx*:  $\{\} \triangleright c : A \implies \models c : A \langle \text{proof} \rangle$

### 6.2.4 Derived rules

**lemma** *WEAK*[*rule-format*]:

$$G \triangleright c : A \implies (\forall H . G \subseteq H \longrightarrow H \triangleright c : A) \langle \text{proof} \rangle$$

**lemma** *CutAux*:

$$\begin{aligned} & (H \triangleright c : A) \implies \\ & (\forall G P D . (H = (\text{insert } P D) \longrightarrow G \triangleright \text{Call} : P \longrightarrow (G \subseteq D) \\ & \quad \longrightarrow D \triangleright c : A)) \langle \text{proof} \rangle \end{aligned}$$

**lemma** *Cut*:  $\llbracket G \triangleright \text{Call} : P ; (\text{insert } P G) \triangleright c : A \rrbracket \implies G \triangleright c : A \langle \text{proof} \rangle$

**definition** *verified*:  $\text{Assn set} \Rightarrow \text{bool}$

**where** *verified*  $G = (\forall A . A : G \longrightarrow G \triangleright \text{body} : A)$

**lemma** *verified-preserved*:  $\llbracket \text{verified } G; A : G \rrbracket \implies \text{verified } (G - \{A\}) \langle \text{proof} \rangle \langle \text{proof} \rangle \langle \text{proof} \rangle$

**theorem** *Mutrec*:

$$\llbracket \text{finite } G; \text{card } G = n; \text{verified } G; A : G \rrbracket \implies \{\} \triangleright \text{Call} : A \langle \text{proof} \rangle$$

### 6.2.5 Completeness

**definition** *SSpec*:  $\text{OBJ} \Rightarrow \text{Assn}$

**where** *SSpec*  $c s t = (s, c \Downarrow t)$

**lemma** *SSpec-valid*:  $\models c : (\text{SSpec } c) \langle \text{proof} \rangle$

**lemma** *SSpec-strong*:  $\models c : A \implies \forall s t . \text{SSpec } c s t \longrightarrow A s t \langle \text{proof} \rangle$

**lemma** *SSpec-derivable*:  $G \triangleright \text{Call} : \text{SSpec } \text{Call} \implies G \triangleright c : \text{SSpec } c \langle \text{proof} \rangle$

**definition** *StrongG* ::  $\text{Assn set}$

where  $StrongG = \{SSpec\ Call\}$

**lemma** *StrongG-Body*:  $StrongG \triangleright body : SSpec\ Call\langle proof \rangle$

**lemma** *StrongG-verified*:  $verified\ StrongG\langle proof \rangle$

**lemma** *SSpec-derivable-empty*:  $\{\} \triangleright c : SSpec\ c\langle proof \rangle$

**theorem** *VDM-Complete*:  $\models c : A \implies \{\} \triangleright c : A\langle proof \rangle\langle proof \rangle\langle proof \rangle$

**theory** *PBIJ* imports *OBJ* begin

### 6.3 Partial bijections

Partial bijections between locations will be used in the next section to define indistinguishability of objects and heaps. We define such bijections as sets of pairs which satisfy the obvious condition.

**types**  $PBij = (Location \times Location)\ set$

**definition**  $Pbij :: PBij\ set$

where  $Pbij = \{ \beta . \forall\ l1\ l2\ l3\ l4 . (l1, l2) : \beta \longrightarrow (l3, l4) : \beta \longrightarrow ((l1 = l3) = (l2 = l4)) \}$

Domain and codomain are defined as expected.

**definition**  $Pbij-Dom :: PBij \Rightarrow (Location\ set)$

where  $Pbij-Dom\ \beta = \{ l . \exists\ ll . (l, ll) : \beta \}$

**definition**  $Pbij-Rng :: PBij \Rightarrow (Location\ set)$

where  $Pbij-Rng\ \beta = \{ ll . \exists\ l . (l, ll) : \beta \}$

We also define the inverse operation, the composition, and a test deciding when one bijection extends another.

**definition**  $Pbij-inverse :: PBij \Rightarrow PBij$

where  $Pbij-inverse\ \beta = \{ (l, ll) . (ll, l) : \beta \}\langle proof \rangle\langle proof \rangle$

**definition**  $Pbij-compose :: PBij \Rightarrow PBij \Rightarrow PBij$

where  $Pbij-compose\ \beta\ \gamma = \{ (l, ll) . \exists\ ll1 . (l, ll1) : \beta \wedge (ll1, ll) : \gamma \}\langle proof \rangle\langle proof \rangle$

**definition**  $Pbij-extends :: PBij \Rightarrow PBij \Rightarrow bool$

where  $Pbij-extends\ \gamma\ \beta = (\beta \subseteq \gamma)$

These definitions satisfy the following properties.

**lemma** *Pbij-insert*:

$\llbracket \beta \in Pbij ; l \notin Pbij-Rng\ \beta ; ll \notin Pbij-Dom\ \beta \rrbracket$

$\implies insert\ (ll, l)\ \beta \in Pbij\langle proof \rangle$

**lemma** *Pbij-injective*:

$\beta : Pbij \implies (\forall\ l1\ l2 . (l1, l) : \beta \longrightarrow (l2, l) : \beta \longrightarrow l1 = l2)\langle proof \rangle$

**lemma** *Pbij-inverse-DomRng*[rule-format]:

$\gamma = Pbij-inverse\ \beta \implies$

$(Pbij-Dom\ \beta = Pbij-Rng\ \gamma \wedge Pbij-Dom\ \gamma = Pbij-Rng\ \beta)\langle proof \rangle$

**lemma** *Pbij-inverse-Dom*:  $Pbij-Dom\ \beta = Pbij-Rng\ (Pbij-inverse\ \beta)\langle proof \rangle$

**lemma** *Pbij-inverse-Rng*:  $Pbij-Dom\ (Pbij-inverse\ \beta) = Pbij-Rng\ \beta\langle proof \rangle$

**lemma** *Pbij-inverse-Pbij*:  $\beta : Pbij \implies (Pbij-inverse\ \beta) : Pbij\langle proof \rangle$

**lemma** *Pbij-inverse-Inverse*[rule-format]:  
 $\gamma = \text{Pbij-inverse } \beta \implies (\forall l \ ll . ((l, ll):\beta) = ((ll, l):\gamma)) \langle \text{proof} \rangle$

**lemma** *Pbij-compose-Dom*:  
 $\text{Pbij-Dom } (\text{Pbij-compose } \beta \ \gamma) \subseteq \text{Pbij-Dom } \beta \langle \text{proof} \rangle$

**lemma** *Pbij-compose-Rng*:  
 $\text{Pbij-Rng } (\text{Pbij-compose } \beta \ \gamma) \subseteq \text{Pbij-Rng } \gamma \langle \text{proof} \rangle$

**lemma** *Pbij-compose-Pbij*:  
 $\llbracket \beta : \text{Pbij}; \gamma : \text{Pbij} \rrbracket \implies \text{Pbij-compose } \beta \ \gamma : \text{Pbij} \langle \text{proof} \rangle$

**lemma** *Pbij-extends-inverse*:  
 $\text{Pbij-extends } \gamma \ (\text{Pbij-inverse } \beta) = \text{Pbij-extends } (\text{Pbij-inverse } \gamma) \ \beta \langle \text{proof} \rangle$

**lemma** *Pbij-extends-reflexive*:  $\text{Pbij-extends } \beta \ \beta \langle \text{proof} \rangle$

**lemma** *Pbij-extends-transitive*:  
 $\llbracket \text{Pbij-extends } \beta \ \gamma; \text{Pbij-extends } \gamma \ \delta \rrbracket \implies \text{Pbij-extends } \beta \ \delta \langle \text{proof} \rangle \langle \text{proof} \rangle$

**lemma** *Pbij-inverse-extends-twice*:  
 $\text{Pbij-extends } (\text{Pbij-inverse } (\text{Pbij-inverse } \beta)) \ \beta \langle \text{proof} \rangle$

The identity bijection on a heap associates each element of the heap's domain with itself.

**definition** *mkId*:: $\text{Heap} \Rightarrow (\text{Location} \times \text{Location}) \text{ set}$   
**where**  $\text{mkId } h = \{(l1, l2) . l1 = l2 \wedge l1 : \text{Dom } h\}$

**lemma** *mkId1*:  $(\text{mkId } h): \text{Pbij} \langle \text{proof} \rangle$   
**lemma** *mkId2*:  $\text{Pbij-Dom } (\text{mkId } h) = \text{Dom } h \langle \text{proof} \rangle$   
**lemma** *mkId2b*:  $\text{Pbij-Rng } (\text{mkId } h) = \text{Dom } h \langle \text{proof} \rangle$   
**lemma** *mkId4*:  $l: \text{Dom } h \implies (l, l): (\text{mkId } h) \langle \text{proof} \rangle$   
**lemma** *mkId4b*:  $(l, ll): (\text{mkId } h) \implies l: \text{Dom } h \wedge l = ll \langle \text{proof} \rangle$

End of theory PBIJ

end

**theory** *VS-OBJ* **imports** *VDM-OBJ* *PBIJ* **begin**

## 6.4 Non-interference

### 6.4.1 Indistinguishability relations

We have the usual two security types.

**datatype** *TP* = *low* | *high*

Global contexts assigns security types to program variables and field names.

**consts** *CONTEXT* :: *Var*  $\Rightarrow$  *TP*  
**consts** *GAMMA* :: *Field*  $\Rightarrow$  *TP*

Indistinguishability of values depends on a partial bijection  $\beta$ .

**inductive-set** *twiddleVal*:: $(\text{PBij} \times \text{Val} \times \text{Val}) \text{ set}$   
**where**

*twiddleVal-Null*:  $(\beta, RVal\ Nullref, RVal\ Nullref) : twiddleVal$

| *twiddleVal-Loc*:  $(l1, l2) : \beta \implies$   
 $(\beta, RVal\ (Loc\ l1), RVal\ (Loc\ l2)) : twiddleVal$   
 | *twiddleVal-IVal*:  $i1 = i2 \implies (\beta, IVal\ i1, IVal\ i2) : twiddleVal$

For stores, indistinguishability is as follows.

**definition** *twiddleStore*:: $PBij \Rightarrow Store \Rightarrow Store \Rightarrow bool$   
**where** *twiddleStore*  $\beta\ s1\ s2 =$   
 $(\forall x. CONTEXT\ x = low \longrightarrow (\beta, s1\ x, s2\ x) : twiddleVal)$

**abbreviation** *twiddleStore-syntax*  $(- \approx_{\beta} - [100,100] 50)$   
**where**  $s \approx_{\beta} t == twiddleStore\ \beta\ s\ t$

On objects, we require the values in low fields to be related, and the sets of defined low fields to be equal.

**definition** *LowDom*:: $((Field \times Val)\ list) \Rightarrow Field\ set$   
**where** *LowDom*  $F = \{f . \exists v . lookup\ F\ f = Some\ v \wedge GAMMA\ f = low\}$

**definition** *twiddleObj*:: $PBij \Rightarrow Object \Rightarrow Object \Rightarrow bool$   
**where** *twiddleObj*  $\beta\ o1\ o2 = ((fst\ o1 = fst\ o2) \wedge$   
 $LowDom\ (snd\ o1) = LowDom\ (snd\ o2) \wedge$   
 $(\forall f\ v\ w . lookup\ (snd\ o1)\ f = Some\ v \longrightarrow$   
 $lookup\ (snd\ o2)\ f = Some\ w \longrightarrow$   
 $GAMMA\ f = low \longrightarrow$   
 $(\beta, v, w) : twiddleVal))$

On heaps, we require locations related by  $\beta$  to contain indistinguishable objects. Domain and codomain of the bijection should be subsets of the domains of the heaps, of course.

**definition** *twiddleHeap*:: $PBij \Rightarrow Heap \Rightarrow Heap \Rightarrow bool$   
**where** *twiddleHeap*  $\beta\ h1\ h2 = (\beta : Pbij \wedge$   
 $Pbij-Dom\ \beta \subseteq Dom\ h1 \wedge$   
 $Pbij-Rng\ \beta \subseteq Dom\ h2 \wedge$   
 $(\forall l\ ll\ v\ w . (l, ll) : \beta \longrightarrow$   
 $lookup\ h1\ l = Some\ v \longrightarrow$   
 $lookup\ h2\ ll = Some\ w \longrightarrow$   
 $twiddleObj\ \beta\ v\ w))$

We also define a predicate which expresses when a state does not contain dangling pointers.

**definition** *noLowDPs*:: $State \Rightarrow bool$   
**where** *noLowDPs*  $S = (case\ S\ of\ (s, h) \Rightarrow$   
 $(\forall x\ l . CONTEXT\ x = low \longrightarrow s\ x = RVal\ (Loc\ l) \longrightarrow l : Dom\ h) \wedge$   
 $(\forall ll\ c\ F\ fl . lookup\ h\ ll = Some\ (c, F) \longrightarrow GAMMA\ f = low \longrightarrow$   
 $lookup\ F\ f = Some\ (RVal\ (Loc\ l)) \longrightarrow l : Dom\ h)))$

The motivation for introducing this notion stems from the intended interpretation of the proof rule for skip, where the initial and final states

should be low equivalent. However, in the presence of dangling pointers, indistinguishability does not hold as such a dangling pointer is not in the expected bijection  $mkId$ . In contrast, for the notion of indistinguishability we use (see the following definition), reflexivity indeed holds, as proven in lemma *twiddle-mkId* below. As a small improvement in comparison to our paper, we now allow dangling pointers in high variables and high fields since these are harmless.

**definition**  $twiddle::PBij \Rightarrow State \Rightarrow State \Rightarrow bool$   
**where**  $twiddle \beta s t = (noLowDPs s \wedge noLowDPs t \wedge (fst s) \approx_{\beta} (fst t) \wedge twiddleHeap \beta (snd s) (snd t))$

**abbreviation**  $twiddle-syntax (- \equiv_{\beta} - [100,100] 50)$   
**where**  $s \equiv_{\beta} t == twiddle \beta s t$

The following properties are easily proven by unfolding the definitions.

**lemma**  $twiddleHeap-isPBij:twiddleHeap \beta h hh \Longrightarrow \beta:PBij\langle proof \rangle$

**lemma**  $isPBij:s \equiv_{\beta} t \Longrightarrow \beta:PBij\langle proof \rangle$

**lemma**  $twiddleVal-inverse:$

$(\beta, w, v) \in twiddleVal \Longrightarrow (Pbij-inverse \beta, v, w) \in twiddleVal\langle proof \rangle$

**lemma**  $twiddleStore-inverse: s \approx_{\beta} t \Longrightarrow t \approx_{(Pbij-inverse \beta)} s\langle proof \rangle$

**lemma**  $twiddleHeap-inverse:$

$twiddleHeap \beta s t \Longrightarrow twiddleHeap (Pbij-inverse \beta) t s\langle proof \rangle$

**lemma**  $Pbij-inverse-twiddle: \llbracket s \equiv_{\beta} t \rrbracket \Longrightarrow t \equiv_{(Pbij-inverse \beta)} s\langle proof \rangle$

**lemma**  $twiddleVal-betaExtend[rule-format]:$

$(\beta, v, w):twiddleVal \Longrightarrow \forall \gamma. Pbij-extends \gamma \beta \longrightarrow (\gamma, v, w):twiddleVal\langle proof \rangle$

**lemma**  $twiddleObj-betaExtend[rule-format]:$

$\llbracket twiddleObj \beta o1 o2; Pbij-extends \gamma \beta \rrbracket \Longrightarrow twiddleObj \gamma o1 o2\langle proof \rangle$

**lemma**  $twiddleVal-compose:$

$\llbracket (\beta, v, u) \in twiddleVal; (\gamma, u, w) \in twiddleVal \rrbracket \Longrightarrow (Pbij-compose \beta \gamma, v, w) \in twiddleVal\langle proof \rangle$

**lemma**  $twiddleHeap-compose:$

$\llbracket twiddleHeap \beta h1 h2; twiddleHeap \gamma h2 h3; \beta \in Pbij; \gamma \in Pbij \rrbracket \Longrightarrow twiddleHeap (Pbij-compose \beta \gamma) h1 h3\langle proof \rangle$

**lemma**  $twiddleStore-compose:$

$\llbracket s \approx_{\beta} r; r \approx_{\gamma} t \rrbracket \Longrightarrow s \approx_{(Pbij-compose \beta \gamma)} t\langle proof \rangle$

**lemma**  $twiddle-compose:$

$\llbracket s \equiv_{\beta} r; r \equiv_{\gamma} t \rrbracket \Longrightarrow s \equiv_{(Pbij-compose \beta \gamma)} t\langle proof \rangle$

**lemma**  $twiddle-mkId: noLowDPs (s, h) \Longrightarrow (s, h) \equiv_{(mkId h)} (s, h)\langle proof \rangle$

We call expressions (semantically) low if the following predicate is satisfied. In particular, this means that if  $e$  evaluates in  $s$  (respectively,  $t$ ) to some location  $l$ , then  $l \in Pbij\_dom(\beta)$  ( $l \in Pbij\_cod(\beta)$ ) holds.

**definition**  $Expr-low::Expr \Rightarrow bool$

**where**  $Expr-low e = (\forall s t \beta. s \approx_{\beta} t \longrightarrow (\beta, evalE e s, evalE e t):twiddleVal)$

A similar notion is defined for boolean expressions, but the fact that these evaluate to (meta-logical) boolean values allows us to replace indistinguishability by equality.

**definition**  $BExpr\text{-low}::BExpr \Rightarrow bool$   
**where**  $BExpr\text{-low } b = (\forall s t \beta . s \approx_\beta t \longrightarrow evalB b s = evalB b t)$

## 6.4.2 Definition and characterisation of security

Now, the notion of security, as defined in the paper. Banerjee and Naumann's paper [1] and the Mobius Deliverable 2.3 [3] contain similar notions.

**definition**  $secure::OBJ \Rightarrow bool$   
**where**  $secure c = (\forall s ss t tt \beta .$   
 $s \equiv_\beta ss \longrightarrow (s, c \Downarrow t) \longrightarrow (ss, c \Downarrow tt) \longrightarrow$   
 $(\exists \gamma . t \equiv_\gamma tt \wedge Pbij\text{-extends } \gamma \beta))$   
 $\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

The type of invariants  $\Phi$  includes a component that holds a partial bijection.

**types**  $TT = (State \times State \times PBij) \Rightarrow bool$

The operator constructing an assertion from an invariant.

**definition**  $Sec :: TT \Rightarrow Assn$   
**where**  $Sec \Phi s t =$   
 $((\forall r \beta . s \equiv_\beta r \longrightarrow \Phi(t, r, \beta)) \wedge$   
 $(\forall r \beta . \Phi(r, s, \beta) \longrightarrow (\exists \gamma . r \equiv_\gamma t \wedge Pbij\text{-extends } \gamma \beta)))$

The lemmas proving that the operator ensures security, and that secure programs satisfy an assertion formed by the operator, are proven in a similar way as in Section 3.

**lemma**  $Prop1A: \models c : Sec \Phi \Longrightarrow secure c \langle proof \rangle$   
**lemma**  $Prop1B:$   
 $secure c \Longrightarrow \models c : Sec (\lambda (r, t, \beta) . \exists s . s , c \Downarrow r \wedge s \equiv_\beta t) \langle proof \rangle$   
**lemma**  $Prop1BB: secure c \Longrightarrow \exists \Phi . \models c : Sec \Phi \langle proof \rangle$   
**lemma**  $Prop1:$   
 $secure c = (\models c : Sec (\lambda (r, t, \beta) . \exists s . (s , c \Downarrow r) \wedge s \equiv_\beta t)) \langle proof \rangle$

## 6.5 Derivation of proof rules

### 6.5.1 Low proof rules

$\langle proof \rangle$   
**lemma**  $SKIP: G \triangleright Skip : Sec (\lambda (s, t, \beta) . s \equiv_\beta t) \langle proof \rangle$   
**lemma**  $ASSIGN:$   
 $Expr\text{-low } e$   
 $\Longrightarrow G \triangleright Assign x e : Sec (\lambda (s, t, \beta) .$   
 $\exists r . s = (update (fst r) x (evalE e (fst r)), snd r)$   
 $\wedge r \equiv_\beta t) \langle proof \rangle \langle proof \rangle$   
**lemma**  $COMP:$   
 $\llbracket G \triangleright c1 : Sec \Phi; G \triangleright c2 : Sec \Psi \rrbracket$   
 $\Longrightarrow G \triangleright (Comp c1 c2) : Sec (\lambda (s, t, \beta) .$   
 $\exists r . \Phi(r, t, \beta) \wedge (\forall w \gamma . r \equiv_\gamma w \longrightarrow \Psi(s, w, \gamma))) \langle proof \rangle \langle proof \rangle$

**lemma IFF:**

$$\begin{aligned} & \llbracket BExpr\text{-low } b; G \triangleright c1 : Sec \Phi; G \triangleright c2 : Sec \Psi \rrbracket \\ \implies & G \triangleright (Iff\ b\ c1\ c2) : Sec (\lambda (s,t,\beta) . \\ & \quad (evalB\ b\ (fst\ t) \longrightarrow \Phi(s,t,\beta)) \wedge \\ & \quad ((\neg\ evalB\ b\ (fst\ t)) \longrightarrow \Psi(s,t,\beta))) \langle proof \rangle \langle proof \rangle \end{aligned}$$

**lemma NEW:**

$$\begin{aligned} & CONTEXT\ x = low \\ \implies & G \triangleright (New\ x\ C) : Sec (\lambda (s,t,\beta) . \\ & \quad \exists\ l\ r . l \notin Dom\ (snd\ r) \wedge r \equiv_{\beta} t \wedge \\ & \quad s = (update\ (fst\ r)\ x\ (RVal\ (Loc\ l)), \\ & \quad (l,(C,[])) \# (snd\ r))) \langle proof \rangle \end{aligned}$$

**lemma GET:**

$$\begin{aligned} & \llbracket CONTEXT\ y = low; GAMMA\ f = low \rrbracket \\ \implies & G \triangleright Get\ x\ y\ f : Sec (\lambda (s,t,\beta) . \\ & \quad \exists\ r\ l\ C\ Flds\ v . (fst\ r)\ y = RVal(Loc\ l) \wedge \\ & \quad lookup\ (snd\ r)\ l = Some(C,Flds) \wedge \\ & \quad lookup\ Flds\ f = Some\ v \wedge r \equiv_{\beta} t \wedge \\ & \quad s = (update\ (fst\ r)\ x\ v,\ snd\ r)) \langle proof \rangle \end{aligned}$$

**lemma PUT:**

$$\begin{aligned} & \llbracket CONTEXT\ x = low; GAMMA\ f = low; Expr\text{-low } e \rrbracket \\ \implies & G \triangleright Put\ x\ f\ e : Sec (\lambda (s,t,\beta) . \\ & \quad \exists\ r\ l\ C\ Flds . (fst\ r)\ x = RVal(Loc\ l) \wedge r \equiv_{\beta} t \wedge \\ & \quad lookup\ (snd\ r)\ l = Some(C,Flds) \wedge \\ & \quad s = (fst\ r, \\ & \quad (l,(C,(f,evalE\ e\ (fst\ r)) \# Flds)) \# (snd\ r))) \langle proof \rangle \end{aligned}$$

Again, we define a fixed point operator over invariants.

**definition FIX::**( $TT \Rightarrow TT$ )  $\Rightarrow TT$

**where**  $FIX\ \varphi = (\lambda (s,t,\beta) .$

$$\forall\ \Phi . (\forall\ ss\ tt\ \gamma . \varphi\ \Phi\ (ss,\ tt,\ \gamma) \longrightarrow \Phi\ (ss,\ tt,\ \gamma)) \longrightarrow \Phi\ (s,\ t,\ \beta))$$

**definition Monotone::**( $TT \Rightarrow TT$ )  $\Rightarrow bool$

**where**  $Monotone\ \varphi =$

$$\begin{aligned} & (\forall\ \Phi\ \Psi . (\forall\ s\ t\ \beta . \Phi(s,t,\beta) \longrightarrow \Psi(s,t,\beta)) \longrightarrow \\ & \quad (\forall\ s\ t\ \beta . \varphi\ \Phi\ (s,t,\beta) \longrightarrow \varphi\ \Psi\ (s,t,\beta))) \end{aligned}$$

$\langle proof \rangle \langle proof \rangle$

For monotone transformers, the construction indeed yields a fixed point.

**lemma Fix-lemma:**  $Monotone\ \varphi \implies \varphi\ (FIX\ \varphi) = FIX\ \varphi \langle proof \rangle$

The operator used in the while rule is defined by

**definition PhiWhileOp::** $BExpr \Rightarrow TT \Rightarrow TT \Rightarrow TT$

**where**  $PhiWhileOp\ b\ \Phi = (\lambda\ \Psi . (\lambda\ (s,\ t,\ \beta) .$

$$\begin{aligned} & (evalB\ b\ (fst\ t) \longrightarrow \\ & \quad (\exists\ r . \Phi\ (r,\ t,\ \beta) \wedge (\forall\ w\ \gamma . r \equiv_{\gamma} w \longrightarrow \Psi(s,\ w,\ \gamma)))) \wedge \\ & (\neg\ evalB\ b\ (fst\ t) \longrightarrow s \equiv_{\beta} t))) \end{aligned}$$

and is monotone:

**lemma PhiWhileOp-Monotone:**  $Monotone\ (PhiWhileOp\ b\ \Phi) \langle proof \rangle$

Hence, we can define its fixed point:

**definition**  $PhiWhile::BExpr \Rightarrow TT \Rightarrow TT$   
**where**  $PhiWhile\ b\ \Phi = FIX\ (PhiWhileOp\ b\ \Phi)$

The while rule may now be given as follows:

**lemma** *WHILE*:  
 $\llbracket BExpr\text{-low}\ b; G \triangleright c : (Sec\ \Phi) \rrbracket$   
 $\implies G \triangleright (While\ b\ c) : (Sec\ (PhiWhile\ b\ \Phi)) \langle proof \rangle$

Operator *PhiWhile* is itself monotone in  $\Phi$ :

**lemma** *PhiWhileMonotone*:  $Monotone\ (\lambda\ \Phi.\ PhiWhile\ b\ \Phi) \langle proof \rangle$

We now give an alternative formulation using an inductive relation equivalent to *FIX*. First, the definition of the variant.

**inductive-set**  $var::(BExpr \times TT \times PBij \times State \times State)\ set$

**where**

$varFalse: \llbracket \neg\ evalB\ b\ (fst\ t); s \equiv_{\beta}\ t \rrbracket \implies (b, \Phi, \beta, s, t) : var$

|  $varTrue:$

$\llbracket evalB\ b\ (fst\ t); \Phi(r, t, \beta); \forall\ w\ \gamma.\ r \equiv_{\gamma}\ w \longrightarrow (b, \Phi, \gamma, s, w) : var \rrbracket$   
 $\implies (b, \Phi, \beta, s, t) : var$

The equivalence of the invariant with the fixed point construction.

$\langle proof \rangle \langle proof \rangle$

**lemma** *varFIXvar*:  $(PhiWhile\ b\ \Phi\ (s, t, \beta)) = ((b, \Phi, \beta, s, t) : var) \langle proof \rangle \langle proof \rangle \langle proof \rangle$

Using this equivalence we can derive the while rule given in our paper from *WHILE*.

**lemma** *WHILE-IND*:

$\llbracket BExpr\text{-low}\ b; G \triangleright c : (Sec\ \Phi) \rrbracket$   
 $\implies G \triangleright While\ b\ c : (Sec\ (\lambda(s, t, \gamma).\ (b, \Phi, \gamma, s, t) : var)) \langle proof \rangle$

We can also show the following property.

$\langle proof \rangle$

**lemma** *var-Monotone*:

$Monotone\ (\lambda\ \Phi.\ (\lambda\ (s, t, \beta).\ (b, \Phi, \beta, s, t) : var)) \langle proof \rangle \langle proof \rangle$

The call rule is formulated for an arbitrary fixed point of a monotone transformer.

**lemma** *CALL*:

$\llbracket (\{Sec\ (FIX\ \varphi)\} \cup X) \triangleright body : Sec\ (\varphi\ (FIX\ \varphi)); Monotone\ \varphi \rrbracket$   
 $\implies X \triangleright Call : Sec\ (FIX\ \varphi) \langle proof \rangle$

## 6.5.2 High proof rules

**definition** *HighSec::Assn*

**where**  $HighSec = (\lambda\ s\ t.\ noLowDPs\ s \longrightarrow s \equiv_{(mkId\ (snd\ s))}\ t)$

**lemma** *CAST*[rule-format]:

$$G \triangleright c : HighSec \implies G \triangleright c : Sec (\lambda (s, t, \beta) . s \equiv_{\beta} t) \langle proof \rangle$$

**lemma** *SkipHigh*:  $G \triangleright Skip : HighSec \langle proof \rangle$

We define a predicate expressing when locations obtained by evaluating an expression are non-dangling.

**definition** *Expr-good*:: $Expr \Rightarrow State \Rightarrow bool$

**where** *Expr-good*  $e s =$

$$(\forall l . evalE e (fst s) = RVal(Loc l) \longrightarrow l : Dom (snd s))$$

**lemma** *AssignHigh*:

$$\begin{aligned} & \llbracket CONTEXT x = high; \forall s . noLowDPs s \longrightarrow Expr-good e s \rrbracket \\ & \implies G \triangleright Assign x e : HighSec \langle proof \rangle \end{aligned}$$

**lemma** *NewHigh*:

$$CONTEXT x = high \implies G \triangleright New x C : HighSec \langle proof \rangle$$

**lemma** *GetHigh*:

$$\llbracket CONTEXT x = high \rrbracket \implies G \triangleright Get x y f : HighSec \langle proof \rangle$$

**lemma** *PutHigh*:

$$\begin{aligned} & \llbracket GAMMA f = high; \forall s . noLowDPs s \longrightarrow Expr-good e s \rrbracket \\ & \implies G \triangleright Put x f e : HighSec \langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle \end{aligned}$$

**lemma** *CompHigh*:

$$\llbracket G \triangleright c : HighSec; G \triangleright d : HighSec \rrbracket \implies G \triangleright Comp c d : HighSec \langle proof \rangle$$

**lemma** *IfHigh*:

$$\llbracket G \triangleright c : HighSec; G \triangleright d : HighSec \rrbracket \implies G \triangleright Iff b c d : HighSec \langle proof \rangle$$

**lemma** *WhileHigh*:  $\llbracket G \triangleright c : HighSec \rrbracket \implies G \triangleright While b c : HighSec \langle proof \rangle$

**lemma** *CallHigh*:

$$(\{HighSec\} \cup G) \triangleright body : HighSec \implies G \triangleright Call : HighSec \langle proof \rangle$$

We combine all rules to an inductive derivation system.

**inductive-set** *Deriv*:: $(Assn\ set \times OBJ \times Assn)\ set$

**where**

*D-CAST*:

$$\begin{aligned} & (G, c, HighSec) : Deriv \implies \\ & (G, c, Sec (\lambda (s, t, \beta) . s \equiv_{\beta} t)) : Deriv \end{aligned}$$

| *D-SKIP*:  $(G, Skip, Sec (\lambda (s, t, \beta) . s \equiv_{\beta} t)) : Deriv$

| *D-ASSIGN*:

$$\begin{aligned} & Expr-low e \implies \\ & (G, Assign x e, Sec (\lambda (s, t, \beta) . \\ & \quad \exists r . s = (update (fst r) x (evalE e (fst r)), snd r) \\ & \quad \wedge r \equiv_{\beta} t)) : Deriv \end{aligned}$$

| *D-COMP*:

$$\begin{aligned} & \llbracket (G, c1, Sec \Phi) : Deriv; (G, c2, Sec \Psi) : Deriv \rrbracket \implies \\ & (G, Comp c1 c2, Sec (\lambda (s, t, \beta) . \\ & \quad \exists r . \Phi(r, t, \beta) \wedge \\ & \quad (\forall w \gamma . r \equiv_{\gamma} w \longrightarrow \Psi(s, w, \gamma))) : Deriv \end{aligned}$$

| *D-IFF*:  
 $\llbracket BExpr\text{-low } b; (G, c1, Sec \Phi):Deriv; (G, c2, Sec \Psi):Deriv \rrbracket \implies$   
 $(G, Iff\ b\ c1\ c2, Sec (\lambda (s,t,\beta) .$   
 $\quad (evalB\ b\ (fst\ t) \longrightarrow \Phi(s,t,\beta)) \wedge$   
 $\quad ((\neg\ evalB\ b\ (fst\ t)) \longrightarrow \Psi(s,t,\beta))) : Deriv$

| *D-NEW*:  
 $CONTEXT\ x = low \implies$   
 $(G, New\ x\ C, Sec (\lambda (s,t,\beta) .$   
 $\quad \exists\ l\ r . l \notin Dom\ (snd\ r) \wedge r \equiv_{\beta} t \wedge$   
 $\quad s = (update\ (fst\ r)\ x\ (RVal\ (Loc\ l)),$   
 $\quad (l,(C,[])) \# (snd\ r))) : Deriv$

| *D-GET*:  
 $\llbracket CONTEXT\ y = low; GAMMA\ f = low \rrbracket \implies$   
 $(G, Get\ x\ y\ f, Sec (\lambda (s,t,\beta) .$   
 $\quad \exists\ r\ l\ C\ Flds\ v. (fst\ r)\ y = RVal(Loc\ l) \wedge$   
 $\quad lookup\ (snd\ r)\ l = Some(C,Flds) \wedge$   
 $\quad lookup\ Flds\ f = Some\ v \wedge r \equiv_{\beta} t \wedge$   
 $\quad s = (update\ (fst\ r)\ x\ v, snd\ r)) : Deriv$

| *D-PUT*:  
 $\llbracket CONTEXT\ x = low; GAMMA\ f = low; Expr\text{-low } e \rrbracket \implies$   
 $(G, Put\ x\ f\ e, Sec (\lambda (s,t,\beta) .$   
 $\quad \exists\ r\ l\ C\ F\ h. (fst\ r)\ x = RVal(Loc\ l) \wedge r \equiv_{\beta} t \wedge$   
 $\quad lookup\ (snd\ r)\ l = Some(C,F) \wedge$   
 $\quad h = (l,(C,(f,evalE\ e\ (fst\ r)) \# F)) \# (snd\ r) \wedge$   
 $\quad s = (fst\ r, h)) : Deriv$

| *D-WHILE*:  
 $\llbracket BExpr\text{-low } b; (G, c, Sec \Phi):Deriv \rrbracket$   
 $\implies (G, While\ b\ c, Sec (\PhiWhile\ b\ \Phi)) : Deriv$

| *D-CALL*:  
 $\llbracket (\{Sec\ (FIX\ \varphi)\} \cup G, body, Sec (\varphi\ (FIX\ \varphi))) : Deriv; Monotone\ \varphi \rrbracket$   
 $\implies (G, Call, Sec (FIX\ \varphi)) : Deriv$

| *D-SKIP-H*:  $(G, Skip, HighSec) : Deriv$

| *D-ASSIGN-H*:  
 $\llbracket CONTEXT\ x = high; \forall\ s . noLowDPs\ s \longrightarrow Expr\text{-good } e\ s \rrbracket$   
 $\implies (G, Assign\ x\ e, HighSec) : Deriv$

| *D-NEW-H*:  $CONTEXT\ x = high \implies (G, New\ x\ C, HighSec) : Deriv$

| *D-GET-H*:  $CONTEXT\ x = high \implies (G, Get\ x\ y\ f, HighSec) : Deriv$

| *D-PUT-H*:  
 $\llbracket GAMMA\ f = high; \forall\ s . noLowDPs\ s \longrightarrow Expr\text{-good } e\ s \rrbracket$

$\implies (G, \text{Put } x \text{ } f \text{ } e, \text{HighSec}): \text{Deriv}$

| *D-COMP-H*:  
 $\llbracket (G, c, \text{HighSec}): \text{Deriv}; (G, d, \text{HighSec}): \text{Deriv} \rrbracket$   
 $\implies (G, \text{Comp } c \text{ } d, \text{HighSec}): \text{Deriv}$

| *D-IFF-H*:  
 $\llbracket (G, c, \text{HighSec}): \text{Deriv}; (G, d, \text{HighSec}): \text{Deriv} \rrbracket$   
 $\implies (G, \text{Iff } b \text{ } c \text{ } d, \text{HighSec}): \text{Deriv}$

| *D-WHILE-H*:  
 $\llbracket (G, c, \text{HighSec}): \text{Deriv} \rrbracket \implies (G, \text{While } b \text{ } c, \text{HighSec}): \text{Deriv}$

| *D-CALL-H*:  
 $(\{\text{HighSec}\} \cup G, \text{body}, \text{HighSec}): \text{Deriv} \implies (G, \text{Call}, \text{HighSec}): \text{Deriv}$

By construction, all derivations represent legal derivations in the program logic. Here's an explicit lemma to this effect.

**lemma** *Deriv-derivable*:  $(G, c, A): \text{Deriv} \implies G \triangleright c: A(\text{proof})$

## 6.6 Type system

We now give a type system in the style of Volpano et al. and then prove its embedding into the system of derived rules. First, type systems for expressions and boolean expressions. These are similar to the ones in Section 3 but require some side conditions regarding the (semantically modelled) operators.

**definition** *opEGood*::  $(\text{Val} \Rightarrow \text{Val} \Rightarrow \text{Val}) \Rightarrow \text{bool}$   
**where** *opEGood*  $f = (\forall \beta \ v \ v' \ w \ w'. (\beta, v, v') \in \text{twiddleVal} \longrightarrow$   
 $(\beta, w, w') \in \text{twiddleVal} \longrightarrow (\beta, f \ v \ w, f \ v' \ w') \in \text{twiddleVal})$

**definition** *compBGood*::  $(\text{Val} \Rightarrow \text{Val} \Rightarrow \text{bool}) \Rightarrow \text{bool}$   
**where** *compBGood*  $f = (\forall \beta \ v \ v' \ w \ w'. (\beta, v, v') \in \text{twiddleVal} \longrightarrow$   
 $(\beta, w, w') \in \text{twiddleVal} \longrightarrow f \ v \ w = f \ v' \ w')$

**inductive-set** *VS-expr*::  $(\text{Expr} \times \text{TP}) \text{ set}$

**where**

*VS-exprVar*:  $\text{CONTEXT } x = t \implies (\text{varE } x, t) : \text{VS-expr}$

|

*VS-exprVal*:

$\llbracket v = \text{RVal Nullref} \vee (\exists i. v = \text{IVal } i) \rrbracket \implies (\text{valE } v, \text{low}) : \text{VS-expr}$

|

*VS-exprOp*:

$\llbracket (e1, t) : \text{VS-expr}; (e2, t) : \text{VS-expr}; \text{opEGood } f \rrbracket$   
 $\implies (\text{opE } f \ e1 \ e2, t) : \text{VS-expr}$

|

*VS-exprHigh*:  $(e, \text{high}) : \text{VS-expr}$

**inductive-set**  $VS\text{-}Bexpr:: (BExpr \times TP) \text{ set}$

**where**

$VS\text{-}BexprOp:$

$\llbracket (e1,t):VS\text{-}expr; (e2,t):VS\text{-}expr; compBGood f \rrbracket$   
 $\implies (compB f e1 e2,t):VS\text{-}Bexpr$

|

$VS\text{-}BexprHigh: (e,high) : VS\text{-}Bexpr$

Next, the core of the type system, the rules for commands. The second side conditions of rules  $VS\text{-}comAssH$  and  $VS\text{-}comPutH$  could be strengthened to  $\forall s. Expr\_good e s$ .

**inductive-set**  $VS\text{-}com:: (TP \times OBJ) \text{ set}$

**where**

$VS\text{-}comGetL:$

$\llbracket CONTEXT y = low; GAMMA f = low \rrbracket$   
 $\implies (low, Get x y f):VS\text{-}com$

|  $VS\text{-}comGetH: CONTEXT x = high \implies (high, Get x y f):VS\text{-}com$

|  $VS\text{-}comPutL:$

$\llbracket CONTEXT x = low; GAMMA f = low; (e, low) : VS\text{-}expr \rrbracket$   
 $\implies (low, Put x f e):VS\text{-}com$

|  $VS\text{-}comPutH:$

$\llbracket GAMMA f = high; \forall s . noLowDPs s \longrightarrow Expr\text{-}good e s \rrbracket$   
 $\implies (high, Put x f e):VS\text{-}com$

|  $VS\text{-}comNewL:$

$CONTEXT x = low \implies (low, New x c) : VS\text{-}com$

|  $VS\text{-}comNewH:$

$CONTEXT x = high \implies (high, New x C):VS\text{-}com$

|  $VS\text{-}comAssL:$

$\llbracket CONTEXT x = low; (e,low):VS\text{-}expr \rrbracket$   
 $\implies (low, Assign x e) : VS\text{-}com$

|  $VS\text{-}comAssH:$

$\llbracket CONTEXT x = high; \forall s . noLowDPs s \longrightarrow Expr\text{-}good e s \rrbracket$   
 $\implies (high, Assign x e) : VS\text{-}com$

|  $VS\text{-}comSkip: (pc, Skip) : VS\text{-}com$

|  $VS\text{-}comComp:$

$\llbracket (pc,c):VS\text{-}com; (pc,d):VS\text{-}com \rrbracket \implies (pc, Comp c d) : VS\text{-}com$

|  $VS\text{-}comIf:$

$\llbracket (b,pc):VS\text{-}Bexpr; (pc,c):VS\text{-}com; (pc,d):VS\text{-}com \rrbracket$   
 $\implies (pc, Iff b c d):VS\text{-}com$

| *VS-comWhile*:  
 $\llbracket (b,pc):VS\text{-}Bexpr; (pc,c):VS\text{-}com \rrbracket \Longrightarrow (pc, While\ b\ c):VS\text{-}com$

| *VS-comSub*:  $(high,c) : VS\text{-}com \Longrightarrow (low,c):VS\text{-}com$

In order to prove the type system sound, we first define the interpretation of expression typings...

**primrec** *SemExpr*::*Expr*  $\Rightarrow$  *TP*  $\Rightarrow$  *bool*  
**where**  
*SemExpr e low* = *Expr-low e* |  
*SemExpr e high* = *True*

... and show the soundness of the typing rules.

**lemma** *ExprSound*:  $(e,tp):VS\text{-}expr \Longrightarrow SemExpr\ e\ tp\langle proof \rangle$

Likewise for the boolean expressions.

**primrec** *SemBExpr*::*BExpr*  $\Rightarrow$  *TP*  $\Rightarrow$  *bool*  
**where**  
*SemBExpr b low* = *BExpr-low b* |  
*SemBExpr b high* = *True*

**lemma** *BExprSound*:  $(e,tp):VS\text{-}Bexpr \Longrightarrow SemBExpr\ e\ tp\langle proof \rangle$

Using these auxiliary lemmas we can prove the embedding of the type system for commands into the system of derived proof rules, by induction on the typing rules.

**theorem** *VS-com-Deriv*[*rule-format*]:  
 $(t,c):VS\text{-}com \Longrightarrow (t=high \longrightarrow (G, c, HighSec):Deriv) \wedge$   
 $(t=low \longrightarrow (\exists \Phi . (G, c, Sec\ \Phi):Deriv))\langle proof \rangle$

Combining this result with the derivability of the derived proof system and the soundness theorem of the program logic yields non-interference of programs that are low typeable.

**theorem** *VS-SOUND*:  $(low,c):VS\text{-}com \Longrightarrow secure\ c\langle proof \rangle$

End of theory *VS-OBJ*

**end**

**theory** *ContextOBJ* **imports** *VS-OBJ* **begin**

## 6.7 Contextual closure

We first define contexts with multiple holes.

**datatype** *CtxtProg* =  
*Ctxt-Here*

| *Ctxt-Skip*  
| *Ctxt-Assign Var Expr*  
| *Ctxt-New Var Class*  
| *Ctxt-Get Var Var Field*  
| *Ctxt-Put Var Field Expr*  
| *Ctxt-Comp CtxtProg CtxtProg*  
| *Ctxt-If BExpr CtxtProg CtxtProg*  
| *Ctxt-While BExpr CtxtProg*  
| *Ctxt-Call*

The definition of a procedure body with holes.

**consts** *Ctxt-Body::CtxtProg*

Next, the substitution of a command into a context:

**primrec** *Fill::CtxtProg  $\Rightarrow$  OBJ  $\Rightarrow$  OBJ*  
**where**  
*Fill Ctxt-Here J = J* |  
*Fill Ctxt-Skip J = Skip* |  
*Fill (Ctxt-Assign x e) J = Assign x e* |  
*Fill (Ctxt-New x c) J = New x c* |  
*Fill (Ctxt-Get x y f) J = Get x y f* |  
*Fill (Ctxt-Put x f e) J = Put x f e* |  
*Fill (Ctxt-Comp C D) J = Comp (Fill C J) (Fill D J)* |  
*Fill (Ctxt-If b C D) J = Iff b (Fill C J) (Fill D J)* |  
*Fill (Ctxt-While b C) J = While b (Fill C J)* |  
*Fill Ctxt-Call J = Call*

The variables mentioned by an expression:

**primrec** *EVars::Expr  $\Rightarrow$  Var set*  
**where**  
*EVars (varE x) = {x}* |  
*EVars (valE v) = {}* |  
*EVars (opE f e1 e2) = EVars e1  $\cup$  EVars e2*

**primrec** *BVars::BExpr  $\Rightarrow$  Var set*  
**where**  
*BVars (compB f e1 e2) = EVars e1  $\cup$  EVars e2*

The variables possibly read from during the evaluation of  $I$  are denoted by  $Vars I$ .

**primrec** *Vars::OBJ  $\Rightarrow$  Var set*  
**where**  
*Vars Skip = {}* |  
*Vars (Assign x e) = EVars e* |  
*Vars (New x C) = {}* |  
*Vars (Get x y f) = {y}* |  
*Vars (Put x f e) = EVars e* |  
*Vars (Comp I J) = Vars I  $\cup$  Vars J* |  
*Vars (While b I) = BVars b  $\cup$  Vars I* |

$Vars (Iff\ b\ I\ J) = BVars\ b \cup Vars\ I \cup Vars\ J \mid$   
 $Vars\ Call = \{\}$   
 $\langle proof \rangle \langle proof \rangle \langle proof \rangle \langle proof \rangle$

An abbreviating definition saying when a value is not a constant location.

**definition**  $ValsNoLoc :: Val \Rightarrow bool$   
**where**  $ValsNoLoc\ v = (v = RVal\ Nullref \vee (\exists\ i . v = IVal\ i))$

Expressions satisfying the following predicate are guaranteed not to return a state-independent location.

**primrec**  $Expr-noLoc :: Expr \Rightarrow bool$   
**where**  
 $Expr-noLoc\ (varE\ x) = True \mid$   
 $Expr-noLoc\ (valE\ v) = ValsNoLoc\ v \mid$   
 $Expr-noLoc\ (opE\ f\ e1\ e2) =$   
 $(Expr-noLoc\ e1 \wedge Expr-noLoc\ e2 \wedge opEGood\ f)$

**primrec**  $BExpr-noLoc :: BExpr \Rightarrow bool$   
**where**  
 $BExpr-noLoc\ (compB\ f\ e1\ e2) =$   
 $(Expr-noLoc\ e1 \wedge Expr-noLoc\ e2 \wedge compBGood\ f)$

By induction on  $e$  one may show the following three properties.

**lemma**  $Expr-lemma1 [rule-format]:$   
 $Expr-noLoc\ e \longrightarrow EVars\ e \subseteq X \longrightarrow$   
 $(\forall x. x \in X \longrightarrow CONTEXT\ x = low) \longrightarrow Expr-low\ e \langle proof \rangle$

**lemma**  $Expr-lemma2 [rule-format]:$   
 $noLowDPs\ (s, h) \longrightarrow$   
 $EVars\ e \subseteq X \longrightarrow Expr-noLoc\ e \longrightarrow$   
 $(\forall x. x \in X \longrightarrow CONTEXT\ x = low) \longrightarrow$   
 $s \approx_{\beta} t \longrightarrow twiddleHeap\ \beta\ h\ k \longrightarrow$   
 $noLowDPs\ (\lambda y. \text{if } x = y \text{ then } evalE\ e\ s \text{ else } s\ y, h) \langle proof \rangle$

**lemma**  $Expr-lemma3 [rule-format]:$   
 $noLowDPs\ (s, h) \longrightarrow EVars\ e \subseteq X \longrightarrow Expr-noLoc\ e \longrightarrow$   
 $lookup\ h\ l = Some\ (C, Flds) \longrightarrow$   
 $(\forall x. x \in X \longrightarrow CONTEXT\ x = low) \longrightarrow$   
 $s \approx_{\beta} t \longrightarrow twiddleHeap\ \beta\ h\ k \longrightarrow$   
 $noLowDPs\ (s, (l, C, (f, evalE\ e\ s) \# Flds) \# h) \langle proof \rangle$

The first of these can be lifted to boolean expressions.

**lemma**  $BExpr-lemma:$   
 $\llbracket BVars\ b \subseteq X; \forall x. x \in X \longrightarrow CONTEXT\ x = low; BExpr-noLoc\ b \rrbracket \Longrightarrow BExpr-low\ b \langle proof \rangle$

For contexts, we define when a set  $X$  of variables is an upper bound for the variables read from. In addition, the  $noLoc$  condition is imposed on expressions occurring in assignments and field modifications in order to express that if these expressions evaluate to locations then these must stem from lookup operations in the state.

**primrec**  $CtxtVars::Var\ set \Rightarrow CtxtProg \Rightarrow bool$   
**where**  
 $CtxtVars\ X\ Ctxt-Here = True \mid$   
 $CtxtVars\ X\ Ctxt-Skip = True \mid$   
 $CtxtVars\ X\ (Ctxt-Assign\ x\ e) = (EVars\ e \subseteq X \wedge Expr-noLoc\ e) \mid$   
 $CtxtVars\ X\ (Ctxt-New\ x\ c) = True \mid$   
 $CtxtVars\ X\ (Ctxt-Get\ x\ y\ f) = (y : X \wedge GAMMA\ f = low) \mid$   
 $CtxtVars\ X\ (Ctxt-Put\ x\ f\ e) =$   
 $(EVars\ e \subseteq X \wedge CONTEXT\ x = low \wedge Expr-noLoc\ e) \mid$   
 $CtxtVars\ X\ (Ctxt-Comp\ C\ D) = (CtxtVars\ X\ C \wedge CtxtVars\ X\ D) \mid$   
 $CtxtVars\ X\ (Ctxt-If\ b\ C\ D) =$   
 $(BVars\ b \subseteq X \wedge CtxtVars\ X\ C \wedge CtxtVars\ X\ D \wedge BExpr-noLoc\ b) \mid$   
 $CtxtVars\ X\ (Ctxt-While\ b\ C) =$   
 $(BVars\ b \subseteq X \wedge CtxtVars\ X\ C \wedge BExpr-noLoc\ b) \mid$   
 $CtxtVars\ X\ Ctxt-Call = True$

A context is "obviously" low if all accessed variables are (contained in a set  $X$  whose members are) low.

**definition**  $LOW::Var\ set \Rightarrow CtxtProg \Rightarrow bool$   
**where**  $LOW\ X\ C = (CtxtVars\ X\ C \wedge (\forall\ x.\ x : X \longrightarrow CONTEXT\ x = low))$   
 $\langle proof \rangle$

Finally, we obtain the following result by induction on an upper bound on the derivation heights of the two executions of  $Fill\ C\ I$ .

**theorem**  $secureI-secureFillI$ :  
 $\llbracket secure\ I; LOW\ X\ C; LOW\ X\ Ctxt-Body; body = Fill\ Ctxt-Body\ I \rrbracket$   
 $\implies secure\ (Fill\ C\ I) \langle proof \rangle$

For a variable

**consts**  $res::Var$

representing the output of the attacking context, the result specialises to

**theorem**  $SecureForAttackingContext$ :  
 $\llbracket secure\ I; LOW\ X\ C; LOW\ X\ Ctxt-Body; s \equiv_{\beta} ss;$   
 $s, (Fill\ C\ I) \Downarrow t; ss, (Fill\ C\ I) \Downarrow tt; body = Fill\ Ctxt-Body\ I;$   
 $CONTEXT\ res = low \rrbracket$   
 $\implies \exists\ \gamma. (\gamma, (fst\ t)\ res, (fst\ tt)\ res) : twiddleVal \wedge Pbij-extends\ \gamma\ \beta \langle proof \rangle$

End of theory ContextObj

**end**

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