Fun With Functions

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Abstract

This is a collection of cute puzzles of the form “Show that if a function satisfies the following constraints, it must be . . . ” Please add further examples to this collection!

Apart from the one about factorial, they all come from the delightful booklet by Terence Tao [1] but go back to Math Olympiads and similar events.

Please add further examples of this kind, either directly or by sending them to me. Let us make this a growing body of fun!

theory FunWithFunctions imports Complex-Main begin

See [1]. Was first brought to our attention by Herbert Ehler who provided a similar proof.

theorem identity1: fixes f :: nat ⇒ nat
assumes fff: ∃ n. f(f(n)) < f(Suc(n))
shows f(n) = n
⟨proof⟩

See [1]. Possible extension: Should also hold if the range of f is the reals!

lemma identity2: fixes f :: nat ⇒ nat
assumes f(k) = k and k ≥ 2
and f-times: ∀ m n. f(m*n) = f(m)*f(n)
and f-mono: ∀ m n. m<n ⇒ f m < f n
shows f(n) = n
⟨proof⟩

One more from Tao’s booklet. If f is also assumed to be continuous, f x = x + 1 holds for all reals, not only rationals. Extend the proof!

theorem plus1:
fixes f :: real ⇒ real
assumes 0: f 0 = 1 and f-add: ∀ x y. f(x+y+1) = f x + f y
assumes r : ℚ shows f(r) = r + 1
⟨proof⟩
The only total model of a naive recursion equation of factorial on integers
is 0 for all negative arguments. Probably folklore.

\textbf{theorem ifac-neg0:} \textbf{fixes ifac :: int \Rightarrow int}
\textbf{assumes ifac-rec:} \forall i. \text{ifac} \; i \; = \; (\text{if} \; i=0 \; \text{then} \; 1 \; \text{else} \; i \times \text{ifac} \; (i - 1))
\textbf{shows} \; i < 0 \; \implies \; \text{ifac} \; i \; = \; 0
\langle \text{proof} \rangle
\textbf{end}

\textbf{References}