Fun With Tilings
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Abstract
Tilings are defined inductively. It is shown that one form of mutilated chess board cannot be tiled with dominoes, while another one can be tiled with L-shaped tiles.

Sections 1 and 2 are by Paulson and described elsewhere [1]. Section 3 is by Nipkow and formalizes a well-known argument from the literature [2]. Please add further fun examples of this kind!

theory Tilings imports Main begin

1 Inductive Tiling

inductive-set tiling :: 'a set set ⇒ 'a set set
for A :: 'a set set where
empty [simp, intro]: \{\} ∈ tiling A |
Un [simp, intro]: \[ a ∈ A; t ∈ tiling A; a ∩ t = \{\} \] ⇒ a ∪ t ∈ tiling A

lemma tiling-UnI [intro]:
\[ t ∈ tiling A; u ∈ tiling A; t ∩ u = \{\} \] ⇒ t ∪ u ∈ tiling A
⟨proof⟩

lemma tiling-Diff1E:
assumes t−a ∈ tiling A and a ∈ A and a ⊆ t
shows t ∈ tiling A
⟨proof⟩

lemma tiling-finite:
assumes ∨a. a ∈ A ⇒ finite a
shows t ∈ tiling A ⇒ finite t
⟨proof⟩
2 The Mutilated Chess Board Cannot be Tiled by Dominoes

The originator of this problem is Max Black, according to J A Robinson. It was popularized as the Mutilated Checkerboard Problem by J McCarthy.

**inductive-set domino :: (nat × nat) set set where**

**horiz [simp]:** \{ (i, j), (i, Suc j) \} ∈ domino |

vertl [simp]: \{ (i, j), (Suc i, j) \} ∈ domino

**lemma domino-finite: d ∈ domino ⇒ finite d**

⟨proof⟩

**declare tiling-finite[OF domino-finite, simp]**

Sets of squares of the given colour

**definition coloured :: nat ⇒ (nat × nat) set where**

coloured b = \{ (i, j). (i + j) mod 2 = b \}

**abbreviation whites :: (nat × nat) set where**

whites ≡ coloured 0

**abbreviation blacks :: (nat × nat) set where**

blacks ≡ coloured (Suc 0)

Chess boards

**lemma Sigma-Suc1 [simp]:**

\{ 0..< Suc n \} × B = (\{ n \} × B) ∪ (\{ 0..<n \} × B)

⟨proof⟩

**lemma Sigma-Suc2 [simp]:**

A × \{ 0..< Suc n \} = (A × \{ n \}) ∪ (A × \{ 0..<n \})

⟨proof⟩

**lemma dominoes-tile-row [intro!]:** \{ i \} × \{ 0..< 2*n \} ∈ tiling domino

⟨proof⟩

**lemma dominoes-tile-matrix: \{ 0..<m \} × \{ 0..< 2*n \} ∈ tiling domino**

⟨proof⟩

**coloured and Dominoes**

**lemma coloured-insert [simp]:**

coloured b ∩ (insert (i, j) t) =

(if (i + j) mod 2 = b then insert (i, j) (coloured b ∩ t)

else coloured b ∩ t)
proof

lemma domino-singletons:
\( d \in \text{domino} \implies \)
(\( \exists \ i \ j. \ whites \cap d = \{(i, j)\} \) \) \land 
(\( \exists m \ n. \ blacks \cap d = \{(m, n)\} \))

⟨proof⟩

Tilings of dominoes

declare
Int-Un-distrib [simp]
Diff-Int-distrib [simp]

lemma tiling-domino-0-1:
\( t \in \text{tiling \ domino} \implies \)
\( \text{card}(whites \cap t) = \text{card}(blacks \cap t) \)

⟨proof⟩

Final argument is surprisingly complex

theorem gen-mutil-not-tiling:
\( t \in \text{tiling \ domino} \implies \)
\( (i + j) \mod 2 = 0 \implies (m + n) \mod 2 = 0 \implies \)
\( \{(i, j), (m, n)\} \subseteq t \)
\( \implies (t - \{(i,j)\} - \{(m,n)\}) \notin \text{tiling \ domino} \)

⟨proof⟩

Apply the general theorem to the well-known case

theorem mutil-not-tiling:
\( t = \{0..< 2 * \text{Suc} \ m\} \times \{0..< 2 * \text{Suc} \ n\} \)
\( \implies t - \{(0,0)\} - \{(\text{Suc}(2 * m), \text{Suc}(2 * n))\} \notin \text{tiling \ domino} \)

⟨proof⟩

3 The Mutilated Chess Board Can be Tiled by Ls

Remove an arbitrary square from a chess board of size \( 2^n \times 2^n \). The result can be tiled by L-shaped tiles. The four possible L-shaped tiles are obtained by dropping one of the four squares from \( \{(x, y), (x + 1, y), (x, y + 1), (x + 1, y + 1)\} \):

definition L2 (x::nat) (y::nat) = \{(x,y), (x+1,y), (x, y+1)\}
definition L3 (x::nat) (y::nat) = \{(x,y), (x+1,y), (x+1, y+1)\}
definition L0 (x::nat) (y::nat) = \{(x+1,y), (x,y+1), (x+1, y+1)\}
definition L1 (x::nat) (y::nat) = \{(x,y), (x,y+1), (x+1, y+1)\}

All tiles:

definition Ls :: (nat * nat) set set where
Ls \equiv \{ L0 x y | x y. True \} \cup \{ L1 x y | x y. True \} \cup 
\{ L2 x y | x y. True \} \cup \{ L3 x y | x y. True \}

3
lemma LinLs: L0 i j : Ls & L1 i j : Ls & L2 i j : Ls & L3 i j : Ls
  ⟨proof⟩

Square $2^n \times 2^n$ grid, shifted by $i$ and $j$:

definition square2 (n::nat) (i::nat) (j::nat) = \{i..<2^n+i\} \times \{j..<2^n+j\}

lemma in-square2 [simp]:
  (a,b) : square2 n i j \iff i\leq a \land a<2^n+i \land j\leq b \land b<2^n+j
  ⟨proof⟩

lemma square2-Suc: square2 (Suc n) i j =
  square2 n i j \cup square2 n (2^n + i) j \cup square2 n i (2^n + j) \cup
  square2 n (2^n + i) (2^n + j)
  ⟨proof⟩

lemma square2-disj: square2 n i j \cap square2 n x y = {} \iff
  (2^n+i \leq x \lor 2^n+x \leq i) \lor (2^n+j \leq y \lor 2^n+y \leq j) (is A = B)
  ⟨proof⟩

Some specific lemmas:

lemma pos-pow2: (0::nat) < 2^n\{n::nat\}
  ⟨proof⟩

declare nat-zero-less-power-iff [simp del] zero-less-power [simp del]

lemma Diff-insert-if: shows
  B \neq \{} \Longrightarrow a:A \Longrightarrow A - insert a B = (A-B - \{a\}) and
  B \neq \{} \Longrightarrow a:\sim: A \Longrightarrow A - insert a B = A-B
  ⟨proof⟩

lemma DisjI1: A Int B = \{} \Longrightarrow (A-X) Int B = \{}
  ⟨proof⟩

lemma DisjI2: A Int B = \{} \Longrightarrow A Int (B-X) = \{}
  ⟨proof⟩

The main theorem:

theorem Ls-can-tile: i \leq a \Longrightarrow a < 2^n + i \Longrightarrow j \leq b \Longrightarrow b < 2^n + j
  \Longrightarrow square2 n i j - \{(a,b)\} : tiling Ls
  ⟨proof⟩

corollary Ls-can-tile00:
  a < 2^n \Longrightarrow b < 2^n \Longrightarrow square2 n 0 0 - \{(a, b)\} \in tiling Ls
  ⟨proof⟩

end
References
