Lifting Definition Option*

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Abstract

We implemented a command, lift-definition-option, which can be used to easily generate elements of a restricted type \( \{ x :: 'a. P x \} \), provided the definition is of the form \( \lambda y_1 \ldots y_n. \text{if check } y_1 \ldots y_n \text{ then Some (generate } y_1 \ldots y_n :: 'a) \text{ else None and check } y_1 \ldots y_n \Rightarrow P \text{ (generate } y_1 \ldots y_n) \) can be proven.

In principle, such a definition is also directly possible using one invocation of lift-definition. However, then this definition will not be suitable for code-generation. To this end, we automated a more complex construction of Joachim Breitner which is amenable for code-generation, and where the test check \( y_1 \ldots y_n \) will only be performed once. In the automation, one auxiliary type is created, and Isabelle’s lifting- and transfer-package is invoked several times.

Contents

1 Introduction 2

2 Usage and limitations 4

3 Examples 4
   3.1 A simple restricted type without type-parameters . . . . . . 4
   3.2 Examples with type-parameters in the restricted type. . . . 5
   3.3 Example from IsaFoR/CeTA . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
1 Introduction

Often algorithms expect that their input satisfies some specific property \( P \). For example, some algorithms require lists as input which have to be sorted, or numbers which have to be positive, or programs which have to be well-typed, etc. Here, there are at least two approaches how one can reason about these algorithms.

The first approach is to guard all soundness properties of the algorithm by the additional precondition \( P \text{ input} \). So, as an example, a binary search algorithm might take arbitrary lists as input, but only if the input list is sorted, then the result of the search is meaningful. Whereas for binary search, this approach is reasonable, there might be problems that the restriction on the input is even crucial for actually defining the algorithm, since without the restriction the algorithm might be non-terminating, and thus, cannot be easily defined using Isabelle’s function-package [4]. As an example, consider an approximation algorithm for \( \sqrt{x} \), where the algorithm should stop once the current approximation \( y \) satisfies \( |x^2 - y^2| < \delta \). Imagine now, that \( \delta \) is a negative number.

To this end, in the second approach the idea is to declare restricted types, so that the algorithms are only invoked with inputs which satisfy the property \( P \). For example, using Isabelle’s lifting- and transfer-package [3], one can easily define a dedicated type for positive numbers, and a function which accesses the internal number, which is then guaranteed to be positive.

```plaintext
typedef `'a pos-num = { x :: `'a :: linordered-field. x > 0 } morphisms num pos by (rule exI[of - 1], auto)

setup-lifting type-definition-pos-num

lemma num-positive: num x > 0 by (transfer, simp)
```

Using these restricted types, it is often possible to define the desired algorithms where non-termination because of invalid inputs is no longer possible, and where soundness properties can be stated without additional preconditions. E.g., for the approximation algorithm, one takes \( \delta \) as input, and the approximation stops, once \( |x^2 - y^2| < \text{num } \delta \) is satisfied.

One question in the second approach is how to actually generate elements of the restricted type. Although one can perform the following definition,

```plaintext
lift-definition create-pos-1 :: `'a :: linordered-field ⇒ `'a pos-num option is
  λ x. if x > 0 then Some x else None
by auto
```

the problem is that corresponding defining equation `create-pos-1 ?x = map-option pos (if (0::?'a) < ?x then Some ?x else None)` is not amenable for code-generation, as it uses the abstraction function `pos` in a way which is not admitted for code-equations [1, 2].
To overcome this problem, Joachim Breitner proposed the following workaround\(^1\), which requires an additional type definition, and some auxiliary definitions, to in the end define \texttt{create-pos} in a way that is amenable for code-generation.

\begin{verbatim}
typedef 'a num-bit = { (x :: 'a :: linordered-field, b). b → x > 0 } by auto

setup-lifting type-definition-num-bit

lift-definition num-bit-bit :: ('a :: linordered-field) num-bit ⇒ bool is snd .

lift-definition num-bit-num :: ('a :: linordered-field) num-bit ⇒ 'a pos-num is
λ (x, b). if b then x else 42 by auto

lift-definition num-bit :: 'a :: linordered-field ⇒ 'a num-bit is
λ x. if x > 0 then (x, True) else (42, False) by auto

definition create-pos-2 :: 'a :: linordered-field ⇒ 'a pos-num option where
create-pos-2 x ≡ let nb = num-bit x
    in if num-bit-bit nb then Some (num-bit-num nb) else None

lemma create-pos-2: create-pos-2 x = Some p ⇒ num p = x
unfolding create-pos-2-def Let-def by (transfer, simp split: if-splits)

export-code create-pos-2 in Haskell

Breitner’s construction has the advantage that the invariant \((0::'a) < x\) only has to be evaluated once (when invoking \texttt{num-bit}). Hence, the construction allows to create data for types with invariants in an efficient, executable, and canonical way.

In this AFP entry we now turned this canonical way into a dedicated method (\texttt{lift-definition-option}) which automatically generates the types and auxiliary functions of Breitner’s construction. As a result it suffices to write:

\begin{verbatim}

lift-definition-option create-pos :: 'a :: linordered-field ⇒ 'a pos-num option is
λ x :: 'a. if x > 0 then Some x else None
by auto

\end{verbatim}

Afterwards, we can directly generate code.

\begin{verbatim}

export-code create-pos in Haskell

Moreover, we automatically generate two soundness theorems, that the generated number is the intended one: \(create-pos ?x = (if (0::?'a) < ?x then Some (pos ?x) else None)\) and \(create-pos ?x = Some ?r ⇒ num ?r = ?x\). Here, the morphisms from the type-definitions reappear, i.e., \texttt{num} and \texttt{pos} in the example.

\end{verbatim}

\(^1\)http://stackoverflow.com/questions/16273812/working-with-isabelles-code-generator-data-refinement-and-higher-order-functi
2 Usage and limitations

The command \texttt{lift-definition-option} is useful to generate elements of some restricted type (say \texttt{'restricted}) which has been defined as \{x. P x\} for some property \(P\) of type \texttt{'base ⇒ bool}. It expects three arguments, namely

- The name of the definition, e.g., \texttt{create-pos}.
- The type of the definition, which must be of the form \texttt{'a1 ⇒ 'a2 ⇒ 'a-dots ⇒ 'a-n ⇒ 'restricted option}
- The right-hand side of the definition which must be of the shape \(\lambda x_1 x_2 x\text-dots x\text-n. \text{if check } x_1 x_2 x\text-dots x\text-n \text{ then Some (generate } x_1 x_2 x\text-dots x\text-n) \text{ else None}\) where \texttt{generate} is of type \texttt{'a1 ⇒ 'a2 ⇒ 'a-dots ⇒ 'a-n ⇒ 'base option}

After providing the three arguments, a proof of \(\text{check } x_1 x_2 x\text-dots x\text-n \implies P (\text{generate } x_1 x_2 x\text-dots x\text-n)\) has to be provided. Then, code-equations will be derived and registered, and in addition two soundness theorems are generated. These are accessible under the names \texttt{def-name} and \texttt{def-name-Some}, provided that the lifting definition uses \texttt{def-name} as first argument.

Note, that \(P\) is automatically extracted from the type-definition of \texttt{'restricted}. Similarly, the default value (42 in the \texttt{'a pos-num-example}) is generated automatically.

To mention a further limitation besides the strict syntactic structure for the right-hand side, it is sometimes required, to add explicit type-annotations in the right-hand-side and the selector, e.g., the \texttt{'a in \lambda x. \text{if } (0::'a) < x \text{ then Some x else None}}.

end

theory Lifting-Definition-Option-Examples
imports
Lifting-Definition-Option
begin

3 Examples

3.1 A simple restricted type without type-parameters

\begin{verbatim}
typedef restricted = \{ i :: int. i mod 2 = 0\} morphisms base restricted
  by (intro exI[of - 4]) auto
setup-lifting type-definition-restricted

Let us start with just using a sufficient criterion for testing for even numbers, without actually generating them, i.e., where the generator is just the identity function.
\end{verbatim}
lift-definition-option restricted-of-simple :: int ⇒ restricted option is
  \( \lambda x :: \text{int. if } x \in \{0, 2, 4, 6\} \text{ then Some } x \text{ else None} \)
proof –
  fix \( x :: \text{int} \)
  assume \( x \in \{0, 2, 4, 6\} \)
  thus \( x \mod 2 = 0 \) by auto
qed

We can also take several input arguments for the test, and generate a
more complex value.
lift-definition-option restricted-of-many-args :: nat ⇒ int ⇒ bool ⇒ restricted
option is
  \( \lambda x y (b :: \text{bool}). \text{if } \text{int } x + y = 5 \text{ then Some } ((\text{int } x + 1) \ast (y + 1)) \text{ else None} \)
proof –
  fix \( x, y \) and \( b :: \text{bool} \)
  assume \( \text{int } x + y = 5 \)
  from \( \text{arg-cong[OF this, of } \lambda x, x \mod 2]\)
  have \( \text{int } x \mod 2 = 1 \lor y \mod 2 = 1 \) by (auto simp: field-simps)
  hence \( (\text{int } x + 1) \mod 2 = 0 \lor (y + 1) \mod 2 = 0 \) by presburger
  thus \( ((\text{int } x+1) \ast (y + 1)) \mod 2 = 0 \) by auto
qed

No problem to use type parameters.
lift-definition-option restricted-of-poly :: 'b list ⇒ restricted option is
  \( \lambda xs :: \text{'b list}. \text{if length } xs = 2 \text{ then Some } (\text{int } (\text{length } (xs))) \text{ else None} \)
proof –
  fix \( xs :: \text{'b list} \)
  assume \( \text{length } xs = 2 \)
  thus \( \text{int } (\text{length } xs) \mod 2 = 0 \) by auto
qed

3.2 Examples with type-parameters in the restricted type.
typedef 'f restrictedf = { xs :: 'f list. length xs < 3 } morphisms basef restrictedf
  by (intro exI[of - Nil]) auto
setup-lifting type-definition-restrictedf
  It does not matter, if we take the same or different type-parameters in
the lift-definition.
lift-definition-option test1 :: 'g ⇒ nat ⇒ 'g restrictedf option is
  \( \lambda (e :: 'g) x. \text{if } x < 2 \text{ then Some } (\text{replicate } x e) \text{ else None} \)
proof –
  fix \( e :: 'g \) and \( x \)
  show \( x < 2 \implies \text{length } (\text{replicate } x e) < 3 \) by auto
qed

lift-definition-option test2 :: 'f ⇒ nat ⇒ 'f restrictedf option is
  \( \lambda (e :: 'f) x. \text{if } x < 2 \text{ then Some } (\text{replicate } x e) \text{ else None} \)
proof

  fix e :: 'f and x
  show \( x < 2 \Rightarrow \text{length (replicate \( x \) \( e \))} < 3 \) by auto
qed

  Tests with multiple type-parameters.

typedef \( (\alpha, \beta) \text{ restr} = \{ (xs :: 'a list, ys :: 'f list) \mid \text{length} \( xs \) = \text{length} \( ys \) \}\)
morphisms base \( \text{ restr} \)
by (rule exI[of - (\[]), []], auto)

setup-lifting type-definition-restr

lift-definition-option restr-of-pair :: 'g \Rightarrow 'e list \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow ('e, \text{nat}) \text{ restr option}
is \( \lambda (z :: 'g) (xs :: 'e list) (y :: \text{nat}) \cdot \text{if length} \( xs \) = \text{n} \text{ then Some (xs, replicate} \( n \) \( y \)) \text{ else None} \)
proof

  fix xs :: 'e list and y :: \text{nat} and \( n 
  assume \( \ast: \forall fidx \in \text{set} \( s \). (\forall i \in \text{set} (\text{snd fidx}) \cdot i < \text{snd (fst fidx})) \)
  thus case (xs, replicate \( n \) \( y \)) of (xs, ys) \Rightarrow \text{length} \( xs \) = \text{length} \( ys \)
  by auto
qed

3.3 Example from \textit{IsaFoR/CeTA}

An argument filter is a mapping \( \pi \) from \( n \)-ary function symbols into lists of positions, i.e., where each position is between 0 and \( n-1 \). In \textit{IsaFoR}, (Isabelle’s Formalization of Rewriting) and \textit{CeTA} [5], the corresponding certifier for term rewriting related properties, this is modelled as follows, where a partial argument filter in a map is extended to a full one by means of a default filter.

typedef \( ('f af \equiv \{ (\pi :: 'f \times \text{nat} \Rightarrow \text{nat list}) \cdot (\forall f n. \text{set} (\pi (f, n)) \subseteq \{0 ..< n\}) \}\)
morphisms \( \text{af Abs-af} \)
by (rule exI[of - \lambda \cdot \cdot [], auto]

setup-lifting type-definition-af

type-synonym \( ('f af-impl = (('f \times \text{nat}) \times \text{nat list}) \text{list} \)

fun fun-of-map-fun :: ('a \Rightarrow 'b option) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \text{ where}
fun-of-map-fun m f a = (case m a of Some \( b \Rightarrow b \mid \text{None} \Rightarrow f a \))

lift-definition-option af-of :: ('f af-impl \Rightarrow ('f af option) \text{ is}
\lambda s :: ('f af-impl. if (\forall fidx \in \text{set} \( s \). (\forall i \in \text{set} (\text{snd fidx}) \cdot i < \text{snd (fst fidx)}))
then Some (fun-of-map-fun (map-of s) (\lambda (f, n). \{0 ..< n\})) \text{ else None}
proof (intro allI)

  fix pi :: ('f af-impl and \( f n \)
let ?pi = fun-of-map-fun (map-of \( pi \)) (\lambda (f, n). \{0 ..< n\})
assume \( \ast: \forall fidx \in \text{set} \( \pi \). (\forall i \in \text{set} (\text{snd fidx}) \cdot i < \text{snd (fst fidx)}) \)
show set {?pi (f, n)} ⊆ {0..<n}
proof (cases map-of pi (f,n))
  case (Some idx)
    from map-of-SomeD[OF Some] Some
    have ?pi (f,n) = idx ((f, n), idx) ∈ set pi by auto
    with * Some show ?thesis by auto
qed auto
qed

3.4 Code generation tests and derived theorems

export-code
  restricted-of-many-args
  restricted-of-simple
  restricted-of-poly
  test1
  test2
  restr-of-pair
  af-of
in Haskell

thm
  restricted-of-many-args-Some restricted-of-many-args
  restricted-of-simple-Some restricted-of-simple
  restricted-of-poly-Some restricted-of-poly
  test1-Some test1
  test2-Some test2
  restr-of-pair-Some restr-of-pair
  af-of-Some af-of
end

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References


